RESTRICTION BOUNDS FOR THE FREE RESOLVENT AND RESONANCES IN LOSSY SCATTERING

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Abstract. We establish high energy L^2 estimates for the restriction of the free Green's function to hypersurfaces in $\mathbb{R}^d$, $d \geq 1$. As an application, we estimate the size of a logarithmic resonance free region for scattering by potentials of the form $V \otimes \delta_\Gamma$, where $\Gamma \subset \mathbb{R}^d$ is a finite union of compact subsets of embedded hypersurfaces. In odd dimensions we prove a resonance expansion for solutions to the wave equation with such a potential.

1. Introduction

Scattering by potentials is used in math and physics to study waves in many physical systems (see for example [7], [16], [29], [11] and the references therein). Examples include acoustics in concert halls, scattering in open microwave cavities, and scattering of particles by atoms. A case of recent interest is scattering in quantum corrals that are constructed using scanning tunneling microscopes [4], [10]. One model for this system is that of a delta function potential on the boundary of a domain $\Omega \subset \mathbb{R}^d$ (see for example [2], [4], [10]). In this paper we study scattering by a generalization of such delta function potentials on hypersurfaces.

We assume that $\Gamma \subset \mathbb{R}^d$ is a finite union $\Gamma = \bigcup_{j=1}^m \Gamma_j$, where each Γ_j is a compact subset of an embedded $C^{1,1}$ hypersurface; equivalently, by subdividing we may take Γ_j to be a compact subset of the graph of a $C^{1,1}$ function with respect to some coordinate. Here, $C^{1,1}$ is the space of functions whose first derivatives are Lipschitz continuous. The Bunimovich stadium is an example of a domain in two dimensions with boundary that is $C^{1,1}$, but not C^2. Let δ_Γ denote $(d-1)$-dimensional Hausdorff measure on Γ, which on each Γ_j agrees with Lebesgue induced surface measure on Γ_j, and let $L^2(\Gamma)$ be the associated space of square-integrable functions on Γ. Although the compact sets Γ_j and Γ may be irregular, the estimates we need in $L^2(\Gamma)$ will follow from L^2 estimates over the hypersurfaces containing the Γ_j, hence the detailed analysis in this paper will take place on $C^{1,1}$ hypersurfaces. Let γu denote restriction of u to Γ. We take V to be a bounded, self-adjoint operator on $L^2(\Gamma)$, and for $u \in H^1_{\text{loc}}(\mathbb{R}^d)$ define $(V \otimes \delta_\Gamma)u := (V\gamma u)\delta_\Gamma$. Let $-\Delta_{V,\Gamma}$ be the unbounded self-adjoint operator

$$-\Delta_{V,\Gamma} := -\Delta + V \otimes \delta_\Gamma \,.$$

(See Section 2.1 for the formal definition of $-\Delta_{V,\Gamma}$.) We will show that $\sigma_{\text{ess}}(-\Delta_{V,\Gamma}) = [0,\infty)$, and that there are at most a finite number of eigenvalues, each of finite rank, in the interval $(-\infty, 0]$. In contrast to the case of potentials $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ (see [21, Section XIII.13] or [11, Section 3.2]), there may be embedded eigenvalues in $[0,\infty)$, which can arise from the allowed non-local nature of $V \otimes \delta_\Gamma$.

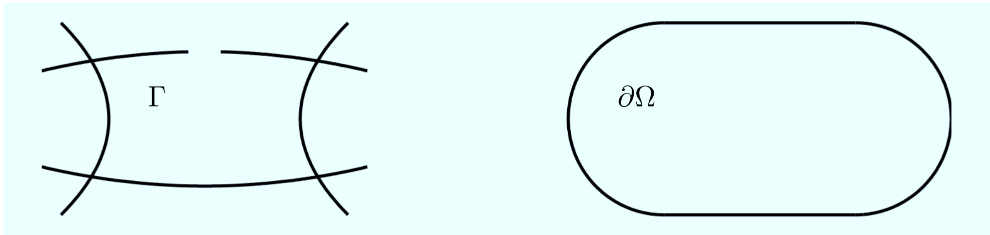

FIGURE 1. Examples of a finite union of compact subsets of strictly convex hypersurfaces, and of the boundary of a domain of $C^{1,1}$ regularity.

Resonances are defined as poles of the meromorphic continuation from $\operatorname{Im}\lambda \gg 1$ of the resolvent

$$R_V(\lambda) = (-\Delta_{V,\Gamma} - \lambda^2)^{-1} .$$

If the dimension d is odd, $R_V(\lambda)$ admits a meromorphic continuation to the entire complex plane, and to the logarithmic covering space of $\mathbb{C} \setminus \{0\}$ if d is even (see Section 6). In even dimensions we will restrict attention to $-\pi \le \arg\lambda \le 2\pi$, so $\operatorname{Im}\lambda > 0$ implies $0 < \arg\lambda < \pi$.

The imaginary part of a resonance gives the decay rate of the corresponding term in the resonance expansion of solutions to the wave equation. Thus, resonances close to the real axis give information about long term behavior of waves. In particular, since the seminal work of Lax-Phillips [16] and Vainberg [28], resonance free regions near the real axis have been used to understand decay of waves.

In this paper, we demonstrate the existence of a resonance free region for $-\Delta_{V,\Gamma}$ on a general class of Γ.

Theorem 1.1. *Let $\Gamma \subset \mathbb{R}^d$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces, and suppose V is a bounded self-adjoint operator on $L^2(\Gamma)$. Then for all $\epsilon > 0$ there exists $R_\epsilon < \infty$ such that, if λ is a resonance for $-\Delta_{V,\Gamma}$, then*

$$\operatorname{Im}\lambda \le -\Big(\tfrac{1}{2} D_\Gamma^{-1} - \epsilon\Big) \log(|\operatorname{Re}\lambda|) \quad \text{if} \quad |\operatorname{Re}\lambda| \ge R_\epsilon , \tag{1}$$

where D_Γ is the diameter of the set Γ. If $d = 1$ then we can replace $\frac{1}{2}$ by 1 in (1). For $d \ge 2$, if Γ can be written as a finite union of compact subsets of strictly convex embedded $C^{2,1}$ hypersurfaces, then we can replace $\frac{1}{2}$ by $\frac{2}{3}$ in (1).

By a strictly convex hypersurface we understand that, with proper choice of normal direction, the second fundamental form of the hypersurface is strictly positive definite, as in the example on the left in Figure 1.

Remarks:

- In case of potential functions $V \in L^\infty_{\text{comp}}(\mathbb{R}^d)$ or $V \in C^\infty_c(\mathbb{R}^d)$, the resonance free region can be improved. For any $M < \infty$ and $\epsilon > 0$, the inequality in (1) can be replaced by, see [17], [18], and [11, Section 3.2]

$$\operatorname{Im}\lambda \le \begin{cases} -\Big(D^{-1}_{\operatorname{supp} V} - \epsilon\Big) \log(|\operatorname{Re}\lambda|) \quad \text{if} \quad |\operatorname{Re}\lambda| \ge R_\epsilon , \quad V \in L^\infty_{\text{comp}}(\mathbb{R}^d) , \\ -M \log(|\operatorname{Re}\lambda|) \quad \text{if} \quad |\operatorname{Re}\lambda| \ge R_M , \quad V \in C^\infty_c(\mathbb{R}^d) . \end{cases}$$

- The bounds on the size of the resonance free region for $-\Delta_{V,\Gamma}$ are not generally optimal, for example in the case that $\Gamma = \partial B(0,1) \subset \mathbb{R}^2$. In [13], the first author uses a microlocal analysis of the transmission problem (8) to obtain sharp bounds in the case that $\Gamma = \partial\Omega$ is C^∞ with Ω strictly convex. In this case, one can replace the constant $\frac{1}{2}$ in (1) by 1 and, under certain nontrapping conditions, one obtains an arbitrarily large logarithmic resonance free region as in the case of $V \in C_c^\infty(\mathbb{R}^d)$.

- For Γ the boundary of a smooth, strictly convex domain $\Omega \subset \mathbb{R}^d$, [9] and [19] studied resonances and local energy decay for the transmission problem with differing wave speeds on Ω and $\mathbb{R}^d \backslash \Omega$, and with prescribed matching conditions at $\partial\Omega$ as opposed to the potential $V \otimes \delta_\Gamma$ considered in this paper.

Let $R_0(\lambda) : H^{-1}_{\rm comp}(\mathbb{R}^d) \to H^1_{\rm loc}(\mathbb{R}^d)$ be the outgoing free resolvent $(-\Delta - \lambda^2)^{-1}$, which is defined by the Fourier multiplier $(|\xi|^2 - \lambda^2)^{-1}$ for $\operatorname{Im}\lambda > 0$, and extended to its domain by analytic continuation in λ; see [11]. In odd dimensions with $d \geq 3$ the domain of $R_0(\lambda)$ is $\mathbb{C}$. If $d = 1$ the domain is $\mathbb{C}\backslash\{0\}$, and $R_0(\lambda)$ has a simple pole at 0. In even dimensions the domain of $R_0(\lambda)$ is the logarithmic cover of $\mathbb{C}\backslash\{0\}$. Theorem 1.1 follows from bounds on the restriction of $R_0(\lambda)$ to Γ. Let $\gamma : H^{\frac{1}{2}+\epsilon}_{\rm loc}(\mathbb{R}^d) \to L^2(\Gamma)$ denote restriction to Γ, and $\gamma^* : L^2(\Gamma) \to H^{-\frac{1}{2}-\epsilon}_{\rm comp}(\mathbb{R}^d)$ its dual, so γ^* is the inclusion map $f \mapsto f\delta_\Gamma$. The trace estimate $\gamma : H^{\frac{1}{2}+\epsilon}_{\rm loc}(\mathbb{R}^d) \to L^2(\Gamma)$ for $\epsilon > 0$ follows from the fact that it holds for subsets Γ_j of $C^{1,1}$ graphs, which is seen by mapping the graph surface to a hyperplane by a $C^{1,1}$ diffeomorphism. For λ in the domain of $R_0(\lambda)$ we then define $G(\lambda) : L^2(\Gamma) \to L^2(\Gamma)$ by restricting the kernel $G_0(\lambda, x, y)$ of $R_0(\lambda)$ to Γ,

$$G(\lambda) := \gamma\, R_0(\lambda)\, \gamma^* .$$

If $d = 1$ then δ_Γ is a finite sum of point measures, and from the formula $G_0(\lambda, x, y) = -(2i\lambda)^{-1} e^{i\lambda|x-y|}$ we see, using the notation of Theorem 1.2 below, that

$$\|G(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\, |\lambda|^{-1}\, e^{D_\Gamma (\operatorname{Im}\lambda)_-} , \qquad d = 1 . \tag{2}$$

In higher dimensions, we establish the following theorem to prove Theorem 1.1.

Theorem 1.2. *Let $\Gamma \subset \mathbb{R}^d$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces. Then $G(\lambda)$ is a compact operator on $L^2(\Gamma)$ for λ in the domain of $R_0(\lambda)$, and there exists C such that*

$$\|G(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} \;\leq\; \begin{cases} C\,\langle\lambda\rangle^{-\frac{1}{2}}\, \log\langle\lambda^{-1}\rangle\, e^{D_\Gamma(\operatorname{Im}\lambda)_-} , & d = 2 , \\ C\,\langle\lambda\rangle^{-\frac{1}{2}}\, \log\langle\lambda\rangle\, e^{D_\Gamma(\operatorname{Im}\lambda)_-} , & d \geq 3 , \end{cases} \tag{3}$$

where D_Γ is the diameter of the set Γ, and we assume $-\pi \leq \arg\lambda \leq 2\pi$ if d is even.

If Γ can be written as a finite union of compact subsets of strictly convex $C^{2,1}$ hypersurfaces, then for some C and all λ in the domain of $R_0(\lambda)$ the following stronger estimate holds

$$\|G(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} \;\leq\; \begin{cases} C\,\langle\lambda\rangle^{-\frac{2}{3}}\, \log\langle\lambda\rangle\, \log\langle\lambda^{-1}\rangle\, e^{D_\Gamma(\operatorname{Im}\lambda)_-} , & d = 2 , \\ C\,\langle\lambda\rangle^{-\frac{2}{3}}\, \log\langle\lambda\rangle\, e^{D_\Gamma(\operatorname{Im}\lambda)_-} , & d \geq 3 . \end{cases} \tag{4}$$

Here we set $\langle\lambda\rangle = (2 + |\lambda|^2)^{\frac{1}{2}}$, and $(\operatorname{Im}\lambda)_- = \max(0, -\operatorname{Im}\lambda)$. Compactness follows easily by Rellich's embedding theorem, or the bounds on $G_0(\lambda, x, y)$ in Section 2.2. The powers $\frac{1}{2}$ and $\frac{2}{3}$ in (3) and (4), respectively, are in general optimal. This follows from the fact that the corresponding estimates for the restriction of eigenfunctions in Section 4 are the best possible. The logarithmic divergence at $\lambda = 0$ for $d = 2$ in both (3) and (4) arises from similar divergence for $R_0(\lambda)$. The factor of $\log\langle\lambda\rangle$ in the estimates, which arises from our method of proof via restriction estimates, is likely not needed. For Γ contained in a hyperplane, the estimate (3) for $d \geq 3$ holds without it, and it does not arise in our direct proof of (3) for $d = 2$. We also expect that estimate (4) holds for subsets of strictly convex $C^{1,1}$ hypersurfaces, but do not pursue that here.

In the case that $\operatorname{Im}\lambda \geq |\lambda|^{\frac{1}{2}}$, respectively $\operatorname{Im}\lambda \geq |\lambda|^{\frac{2}{3}}$, the above bounds can be improved upon.

Theorem 1.3. *Let $\Gamma \subset \mathbb{R}^d$ be a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces. Then there esists C such that for $0 \leq \arg\lambda \leq \pi$,*

$$\|G(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq \begin{cases} C\,\langle \operatorname{Im}\lambda\rangle^{-1}\,\log\langle\lambda^{-1}\rangle, & d=2\,,\\ C\,\langle \operatorname{Im}\lambda\rangle^{-1}\,, & d\geq 3\,. \end{cases}$$

We next use the results above to analyze the long term behavior of waves scattered by the potential $V\otimes\delta_\Gamma$. Theorem 1.1 implies that there are only a finite number of resonances in the set $\operatorname{Im}\lambda > -A$, for any $A<\infty$. We give a resonance expansion in odd dimensions for the wave equation

$$\big(\partial_t^2 - \Delta + V\otimes\delta_\Gamma\big)u = 0\,,\quad u(0,x)=0,\quad \partial_t u(0,x) = g \in L^2_{\mathrm{comp}}\,, \tag{5}$$

with solution given by $U(t)g$, where $U(t)g$ can be expressed as an integral (33) of the resolvent $R_V(\lambda)g$. This is also equivalent to the more standard functional calculus expression $\sqrt{-\Delta_{V,\Gamma}}^{\,-1}\sin(t\sqrt{-\Delta_{V,\Gamma}}\,)g$.

Let $m_R(\lambda)$ be the order of the pole of $R_V(\lambda)$ at λ. We let $\mathcal{D}_N$ be the domain of $(-\Delta_{V,\Gamma})^N$, and define

$$\mathcal{D}_{\mathrm{loc}} = \big\{u\,:\,\chi u\in\mathcal{D}_1 \text{ whenever } \chi\in C_c^\infty(\mathbb{R}^d) \text{ and } \chi=1 \text{ on a neighborhood of } \Gamma\big\}\,.$$

Theorem 1.4. *Let $d\geq 1$ be odd, and assume that $\Gamma\subset\mathbb{R}^d$ is a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces, and that V is a self-adjoint operator on $L^2(\Gamma)$.*

Let $0 > -\mu_1^2 > \cdots > -\mu_K^2$ and $0 < \nu_1^2 < \cdots < \nu_M^2$ be the nonzero eigenvalues of $-\Delta_{V,\Gamma}$, with $\mu_j\,,\nu_j>0$, and $\{\lambda_k\}$ the resonances with $\operatorname{Im}\lambda<0$. Then for any $A>0$ and $g\in L^2_{\mathrm{comp}}$, the solution $U(t)g$ to (5) admits an expansion

$$U(t)g = \sum_{j=1}^K (2\mu_j)^{-1}e^{t\mu_j}\Pi_{\mu_j}g + t\,\Pi_0 g + \mathcal{P}_0 g + \sum_{k=1}^M {\nu_j}^{-1}\sin(t\nu_j)\Pi_{\nu_j}g + \sum_{\operatorname{Im}\lambda_k>-A}\;\sum_{\ell=0}^{m_R(\lambda_k)-1} e^{-it\lambda_k}\,t^\ell\,\mathcal{P}_{\lambda_k,\ell}g + E_A(t)g\,, \tag{6}$$

where Π_{μ_j} and Π_{ν_j} respectively denote the projections onto the $-\mu_j^2$ and ν_j^2 eigenspaces, and Π_0 the projection onto the 0-eigenspace. The maps $\mathcal{P}_{\lambda_k,\ell}$ and $\mathcal{P}_0$ are bounded from $L^2_{\mathrm{comp}}\to\mathcal{D}_{\mathrm{loc}}$.

The operator $E_A(t)\,:\,L^2_{\mathrm{comp}}\to L^2_{\mathrm{loc}}$ has the following property: for any $\chi\in C_c^\infty(\mathbb{R}^d)$ equal to 1 on a neighborhood of Γ, and $N\geq 0$, there exists $T_{A,\chi,N}<\infty$ and $C_{A,\chi,N}<\infty$ so that

$$\|\chi E_A(t)\chi\|_{L^2\to\mathcal{D}_N} \leq C_{A,\chi,N}\,e^{-At}\,,\qquad t>T_{A,\chi,N}\,.$$

We refer to Section 7 for more details on the operators $\mathcal{P}_{\lambda_k,\ell}$ and $\mathcal{P}_0$. The restriction that t be larger than a constant depending on the diameter of χ is necessary to ensure that $\chi E_A(t)\chi g$ has no H^{2N} singularities away from Γ in $\mathrm{supp}(\chi)$, although our argument does not give an optimal value for $T_{A,\chi,N}$.

Under the assumption that $\Gamma = \partial\Omega$ for a bounded open domain $\Omega \subset \mathbb{R}^d$, and that V and $\partial\Omega$ satisfy higher regularity assumptions, for $g \in L^2$ we obtain estimates on $\chi E_A(t)\chi g$ in the spaces

$$\mathcal{E}_N := H^1(\mathbb{R}^d) \cap \big(H^N(\Omega) \oplus H^N(\mathbb{R}^d \setminus \overline{\Omega})\big)\,, \qquad N \geq 1\,.$$

If $\partial\Omega$ is of $C^{1,1}$ regularity, and V is bounded $H^{\frac12}(\partial\Omega) \to H^{\frac12}(\partial\Omega)$, then we show in Section 2 that $\mathcal{D}_1 \subset \mathcal{E}_2$, and convergence in $\mathcal{E}_2$ follows from Theorem 1.4. If $\partial\Omega$ is of C^∞ regularity, and V is bounded $H^s(\partial\Omega) \to H^s(\partial\Omega)$ for every s, then $\mathcal{D}_N$ is a closed subspace of $\mathcal{E}_{2N}$ (see (11) below), and we have the following.

Theorem 1.5. *Suppose that $\Gamma = \partial\Omega$ is C^∞, and that V is a self-adjoint map on $L^2(\partial\Omega)$ which is bounded from $H^s(\partial\Omega) \to H^s(\partial\Omega)$ for all s. Then the operator $E_A(t)$ defined in (6) has the following property: for any $\chi \in C_c^\infty(\mathbb{R}^d)$ equal to 1 on a neighborhood of $\overline{\Omega}$, and $N \geq 1$, there exists $T_{A,\chi,N} < \infty$ and $C_{A,\chi,N} < \infty$ so that*

$$\|\chi E_A(t)\chi\|_{L^2\to\mathcal{E}_N} \leq C_{A,\chi,N}\, e^{-At}\,, \qquad t > T_{A,\chi,N}\,.$$

In addition to describing resonances as poles of the meromorphic continuation of the resolvent, we will give a more concrete description of resonances in Sections 6 and 7. We show in Proposition 6.2 and the comments following it, that if Γ is as in Theorem 1.1 then λ is a resonance if and only if there is a nontrivial λ-outgoing solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ to the equation

$$(-\Delta - \lambda^2 + V \otimes \delta_\Gamma)u = 0\,. \tag{7}$$

Here we say that u is λ-*outgoing* if for some $R < \infty$, and some compactly supported distribution g, we can write

$$u(x) = \big(R_0(\lambda)g\big)(x) \quad \text{for} \quad |x| \geq R\,.$$

In case $d = 1$ this definition needs to be modified for $\lambda = 0$. Noting that for $d = 1$ and $\lambda \neq 0$, a λ-outgoing solution equals $c_{\mathrm{sgn}(x)}e^{i\lambda|x|}$ for $|x| \geq R$, we say u is 0-outgoing when $d = 1$ if u is separately constant on $x \geq R$ and $x \leq -R$, for some $R < \infty$.

In case $\Gamma = \partial\Omega$ for a bounded domain Ω, and $V : H^{\frac12}(\partial\Omega) \to H^{\frac12}(\partial\Omega)$, we show that λ-outgoing solvability of (7) is equivalent to solving the following transmission problem. We remark that the Sobolev spaces $H^s(\partial\Omega)$ are well defined for $|s| \leq 2$ if $\partial\Omega$ is $C^{1,1}$, since these spaces are preserved under $C^{1,1}$ changes of coordinates. Also, since $\mathcal{E}_{2,\mathrm{loc}}(\mathbb{R}^d) \subset H^1_{\mathrm{loc}}(\mathbb{R}^d)$, if $u \in \mathcal{E}_{2,\mathrm{loc}}$ then its trace γu belongs to $H^{\frac12}(\partial\Omega)$.

Proposition 1.6. *Assume Ω is a bounded domain in $\mathbb{R}^d$ with $\partial\Omega$ a $C^{1,1}$ hypersurface, and V is a self-adjoint map on $L^2(\partial\Omega)$ which maps $H^{\frac12}(\partial\Omega) \to H^{\frac12}(\partial\Omega)$. Then λ is a resonance of $-\Delta_{V,\partial\Omega}$ if and only if the following system has a nontrivial solution $u \in \mathcal{E}_{2,\mathrm{loc}}(\mathbb{R}^d)$ such* *that, with* $u|_\Omega = u_1\,,\ u|_{\mathbb{R}^d\setminus\overline{\Omega}} = u_2\,,$

$$
\begin{cases}
(-\Delta-\lambda^2)u_1 = 0 & \text{in } \Omega\\
(-\Delta-\lambda^2)u_2 = 0 & \text{in } \mathbb{R}^d\setminus\overline{\Omega}\\
\partial_\nu u_1 + \partial_{\nu'} u_2 + V\gamma u = 0 & \text{on } \partial\Omega\\
u_2 \text{ is } \lambda\text{-outgoing}
\end{cases}
\tag{8}
$$

Here, ∂_ν *and* $\partial_{\nu'}$ *are respectively the interior and exterior normal derivatives of* u *at* $\partial\Omega$*.*

The outline of this paper is as follows. In Section 2 we present the definition of $-\Delta_{V,\Omega}$ and its domain, as well as some preliminary bounds on the outgoing Green's function $G_0(\lambda,x,y)$. In Section 3 we give a simple proof of Theorem 1.2 for $d=2$. In Section 4 we establish Theorem 1.2 for $\operatorname{Im}\lambda>0$ in all dimensions, deriving the estimates from restriction estimates for eigenfunctions of the Laplacian. We include a proof of the desired restriction estimate for hypersurfaces of regularity $C^{1,1}$, since the result appears new, and prove Theorem 1.3. In Section 5 we complete the proof of Theorem 1.2 for $\operatorname{Im}\lambda\le 0$ using the Phragmén-Lindelöf theorem. In Section 6 we demonstrate the meromorphic continuation of $R_V(\lambda)$, give the proof of Theorem 1.1, relate resonances to solvability of (7) by reduction to an equation on Γ, and prove Proposition 1.6. In Section 7 we give more details on the structure of the meromorphic continuation of $R_V(\lambda)$. We establish mapping bounds for compact cutoffs of $R_V(\lambda)$, and use these to prove Theorems 1.4 and 1.5 by a contour integration argument. In Section 8 we prove a needed transmission property estimate for boundaries of regularity $C^{1,1}$.

2. Preliminaries

2.1. **Determination of $-\Delta_{V,\Gamma}$ and its Domain.** We define the operator $-\Delta_{V,\Gamma}$ using the symmetric quadratic form, with dense domain $H^1(\mathbb{R}^d)\subset L^2(\mathbb{R}^d)$,

$$Q_{V,\Gamma}(u,w) := \langle \nabla u,\nabla w\rangle_{L^2(\mathbb{R}^d)} + \langle V\gamma u,\gamma w\rangle_{L^2(\Gamma)}\,.$$

For Γ a finite union of compact subsets of $C^{1,1}$ hypersurfaces (indeed for Γ a bounded subset of a Lipschitz graph), as a special case of (39) we can bound

$$\|\gamma u\|_{L^2(\Gamma)} \le C\,\|u\|_{L^2}^{\frac12}\|u\|_{H^1}^{\frac12} \le C\,\epsilon\,\|u\|_{H^1} + C\,\epsilon^{-1}\|u\|_{L^2}\,.$$

It follows that there exist $c\,,C>0$ such that

$$|Q_{V,\Gamma}(u,w)| \le C\,\|u\|_{H^1}\|w\|_{H^1} \quad \text{and} \quad c\,\|u\|_{H^1}^2 \le Q_{V,\Gamma}(u,u) + C\|u\|_{L^2}^2\,.$$

By Reed-Simon [20, Theorem VIII.15], $Q_{V,\Gamma}(u,w)$ is determined by a unique self-adjoint operator $-\Delta_{V,\Gamma}$, with domain $\mathcal{D}$ consisting of $u\in H^1$ such that $Q_{V,\Gamma}(u,w)\le C\|w\|_{L^2}$ for all $w\in H^1(\mathbb{R}^d)$. By Rellich's embedding lemma, the potential term is compact relative to H^1. It follows by Weyl's essential spectrum theorem, see [21, Theorem XIII.14], that $\sigma_{\mathrm{ess}}(-\Delta_{V,\Gamma}) = [0,\infty)$. Additonally, there are at most a finite number of eigenvalues in $(-\infty,0]$, each of finite multiplicity.

If $u\in\mathcal{D}$, by the Riesz representation theorem we then have $Q_{V,\Gamma}(u,w)=\langle g,w\rangle$ for some $g\in L^2(\mathbb{R}^d)$, and taking $w\in C_c^\infty(\mathbb{R}^d)$ shows that in the sense of distributions

$$-\Delta u + (V\gamma u)\delta_\Gamma = g\,. \tag{9}$$

Conversely, if $u \in H^1(\mathbb{R}^d)$ and (9) holds for some $g \in L^2(\mathbb{R}^d)$, then by density of $C_c^\infty \subset H^1$ we have $Q_{V,\Gamma}(u,w) = \langle g, w\rangle$ for $w \in H^1(\mathbb{R}^d)$, hence $u \in \mathcal{D}$, and $-\Delta_{V,\Gamma} u$ is given by the left hand side of (9). We thus can define

$$\|u\|_{\mathcal{D}} = \|u\|_{H^1} + \|\Delta_{V,\Gamma} u\|_{L^2}\,,$$

where finiteness of the second term carries the assumption that $\Delta_{V,\Gamma} u \in L^2$.

We set $\mathcal{D}_1 = \mathcal{D}$, and recursively define $\mathcal{D}_N \subset \mathcal{D}_1$ for $N \geq 2$ by the condition $\Delta_{V,\Gamma} u \in \mathcal{D}_{N-1}$. We also recursively define

$$\|u\|_{\mathcal{D}_N} = \|u\|_{H^1} + \|\Delta_{V,\Gamma} u\|_{\mathcal{D}_{N-1}}\,, \qquad N \geq 2\,.$$

Suppose that $\chi \in C_c^\infty(\mathbb{R}^d \setminus \Gamma)$ and that $u \in H^1(\mathbb{R}^d)$ solves (9). Then,

$$\Delta(\chi u) = \chi g + 2\nabla\chi\cdot\nabla u + (\Delta\chi)u \in L^2(\mathbb{R}^d)\,.$$

Hence,

$$\|\chi u\|_{H^2} \leq C_\chi \|u\|_{\mathcal{D}}\,.$$

That is, $\mathcal{D} \subset H^1(\mathbb{R}^d) \cap H^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)$, with continuous inclusion. Similar arguments show that

$$\mathcal{D}_N \subset H^1(\mathbb{R}^d) \cap H^{2N}_{\mathrm{loc}}(\mathbb{R}^d \setminus \Gamma)\,.$$

The behavior of u near Γ may be more singular. For V and Γ as in Theorem 1.1, from (9) and the fact that $(V\gamma u)\delta_\Gamma \in H^{-\frac12-\epsilon}(\mathbb{R}^d)$ for all $\epsilon > 0$, we conclude that $u \in H^{\frac32-\epsilon}(\mathbb{R}^d)$. However, under additional assumptions on V and Γ we can give a full description of $\mathcal{D}$ near Γ.

For the purposes of the remainder of this section we assume that $\Gamma = \partial\Omega$ for some bounded open domain $\Omega \subset \mathbb{R}^d$, and that $\partial\Omega$ is a $C^{1,1}$ hypersurface; that is, locally $\partial\Omega$ can be written as the graph of a $C^{1,1}$ function. We assume also that $V : H^{\frac12}(\partial\Omega) \to H^{\frac12}(\partial\Omega)$. Then since $u \in H^1(\mathbb{R}^d)$, and $\gamma : H^s(\mathbb{R}^d) \to H^{s-\frac12}(\partial\Omega)$ for $s \in (\frac12, 2]$, we have $V\gamma u \in H^{\frac12}(\partial\Omega)$. By (9) we can write u as $(-\Delta)^{-1}g$ plus the single layer potential of a $H^{\frac12}(\partial\Omega)$ function, hence Proposition 8.2 shows that

$$\mathcal{D} \subset \mathcal{E}_2 = H^1(\mathbb{R}^d) \cap (H^2(\Omega) \oplus H^2(\mathbb{R}^d \setminus \overline{\Omega}))\,,$$

with continuous inclusion. We remark that $H^2(\Omega)$ and $H^2(\mathbb{R}^d \setminus \overline{\Omega})$ can be identified as restrictions of $H^2(\mathbb{R}^d)$ functions; see [8] and [23, Theorem VI.5]. Thus, if $u \in \mathcal{D}$ then u has a well defined trace on $\partial\Omega$ of regularity $H^{\frac32}(\partial\Omega)$, and the first derivatives of u have one-sided traces from the interior and exterior, of regularity $H^{\frac12}(\partial\Omega)$.

For $w \in H^1(\mathbb{R}^d)$ and $u \in \mathcal{E}_2$, it follows from Green's identities that

$$Q_{V,\partial\Omega}(u,w) = \langle -\Delta u, w\rangle_{L^2(\Omega)} + \langle -\Delta u, w\rangle_{L^2(\mathbb{R}^d\setminus\overline{\Omega})} + \langle \partial_\nu u + \partial_{\nu'} u + V\gamma u, \gamma w\rangle_{L^2(\partial\Omega)}\,,$$

where ∂_ν and $\partial_{\nu'}$ denote the exterior normal derivatives from Ω and $\mathbb{R}^d\setminus\overline{\Omega}$. Thus, in the case that V is bounded from $H^{\frac12}(\partial\Omega) \to H^{\frac12}(\partial\Omega)$, we can completely characterize the domain $\mathcal{D}$ of the self-adjoint operator $-\Delta_{V,\partial\Omega}$ as

$$\mathcal{D} = \big\{ u \in \mathcal{E}_2 \quad \text{such that} \quad \partial_\nu u + \partial_{\nu'} u + V\gamma u = 0 \big\}\,, \tag{10}$$

in which case $\Delta_{V,\partial\Omega} u = \Delta u|_\Omega \oplus \Delta u|_{\mathbb{R}^d\setminus\overline{\Omega}}$.

If $\partial\Omega$ is a C^∞ hypersurface, and $V : H^s(\partial\Omega) \to H^s(\partial\Omega)$ is bounded for all s, then Proposition 8.1 and induction, as in the proof of Lemma 7.2, show that $\mathcal{D}_N \subset \mathcal{E}_{2N}$. Induction also shows that $\mathcal{D}_N$ can be characterized as the subspace of $\mathcal{E}_{2N}$ consisting of u that satisfy the following matching conditions:

(11)
$$\gamma\big(\Delta^j u|_\Omega\big) = \gamma\big(\Delta^j u|_{\mathbb{R}^d\setminus\overline{\Omega}}\big)\,, \qquad \partial_\nu\big(\Delta^j u|_\Omega\big)+\partial_{\nu'}\big(\Delta^j u|_{\mathbb{R}^d\setminus\overline{\Omega}}\big)+V\gamma\big(\Delta^j u|_\Omega\big) = 0\,, \quad \text{for} \quad 0\le j\le N-1\,.$$

2.2. **Bounds on Green's function.** We conclude this section by reviewing bounds on the convolution kernel $G_0(\lambda,x,y)$ associated to the operator $R_0(\lambda)$. It can be written in terms of the Hankel functions of the first kind,

$$G_0(\lambda,x,y) = C_d\,\lambda^{d-2}\,\big(\lambda|x-y|\big)^{-\frac{d-2}{2}}\,H^{(1)}_{\frac{d}{2}-1}\big(\lambda|x-y|\big)\,,$$

for some constant C_d. If $d\ge 3$ is odd, this can be written as a finite expansion

$$G_0(\lambda,x,y) = \lambda^{d-2}\,e^{i\lambda|x-y|}\sum_{j=\frac{d-1}{2}}^{d-2}\frac{c_{d,j}}{\big(\lambda|x-y|\big)^j}\,.$$

For $x\ne y$ this form extends to $\lambda\in\mathbb{C}$, and defines the analytic extension of $R_0(\lambda)$. In particular, for $d\ge 3$ odd we have the upper bounds

$$|G_0(\lambda,x,y)| \lesssim \begin{cases} |x-y|^{2-d}\,, & |x-y|\le|\lambda|^{-1}\,,\\ e^{-\operatorname{Im}\lambda|x-y|}\,|\lambda|^{\frac{d-3}{2}}\,|x-y|^{\frac{1-d}{2}}\,, & |x-y|\ge|\lambda|^{-1}\,. \end{cases} \tag{12}$$

If $d\ge 4$ is even, the bounds (12) hold for $\operatorname{Im}\lambda>0$, as well as for the analytic extension to $-\pi\le\arg\lambda\le 2\pi$. For $-\pi<\arg\lambda<2\pi$ this follows by the asymptotics of $H^{(1)}_n(z)$; see for example [1, (9.2.3)]. To see that it extends to the closed sector, we use Stone's formula (see [11]),

$$G_0(e^{i\pi}\lambda,x,y) - G_0(\lambda,x,y) = \frac{i}{2}\,\frac{\lambda^{d-2}}{(2\pi)^{d-1}}\int_{\mathbb{S}^{d-1}} e^{i\lambda\langle x-y,\omega\rangle}\,d\omega = C_d\,\lambda^{d-2}\,\big(\lambda|x-y|\big)^{-\frac{d-2}{2}}\,J_{\frac{d}{2}-1}\big(\lambda|x-y|\big)$$

where $e^{i\pi}$ indicates analytic continuation through positive angle π, and where $d\omega$ is surface measure on the unit sphere $\mathbb{S}^{d-1}\subset\mathbb{R}^d$. This holds in all dimensions for $\lambda>0$, and hence for the analytic continuation. The bounds (12) then follow from the asymptotics of $J_n(z)$ and the bounds for $\operatorname{Im}\lambda\ge 0$. We also note as a consequence of the above that, for $\lambda\in\mathbb{R}\setminus\{0\}$, and any sheet of the continuation in even dimensions,

$$G_0(e^{i\pi}\lambda,x,y) - G_0(\lambda,x,y) = \pi i\,(\operatorname{sgn}\lambda)^d\,|\lambda|^{-1}(2\pi)^{-d}\,\widehat{\delta_{\mathbb{S}^{d-1}_\lambda}}(x-y)\,, \tag{13}$$

where $\delta_{\mathbb{S}^{d-1}_\lambda}$ denotes surface measure on the sphere $|\xi|=|\lambda|$ in $\mathbb{R}^d$, and $\hat g(\xi)=\int e^{-i\langle x,\xi\rangle}\,g(x)\,dx$.

If $d=2$, one has the bounds, see [1, (9.1.8)-(9.2.3)],

$$|G_0(\lambda,x,y)| \lesssim \begin{cases} |\log(\lambda|x-y|)\,|\,, & |x-y|\le\frac12|\lambda|^{-1}\,,\\ e^{-\operatorname{Im}\lambda|x-y|}\,|\lambda|^{-\frac12}\,|x-y|^{-\frac12}\,, & |x-y|\ge\frac12|\lambda|^{-1}\,. \end{cases} \tag{14}$$

3. Estimates for $d=2$

In this section we give an elementary proof of estimate (3) of Theorem 1.2 for $d=2$. Indeed, we can prove the following stronger result, which holds on subsets of Lipschitz graphs.

Theorem 3.1. *Suppose that $d=2$, and that Γ is a finite union $\Gamma=\bigcup_j \Gamma_j$ where each Γ_j is a compact subset of a Lipschitz graph. Then for $-\pi \le \arg\lambda \le 2\pi$, with 1-dimensional Hausdorff measure on Γ,*

$$\|G(\lambda)f\|_{L^2(\Gamma)} \le \begin{cases} C\,\langle\lambda\rangle^{-\frac12}\,\log\langle\lambda^{-1}\rangle\,\langle \operatorname{Im}\lambda\rangle^{-\frac12}\,\|f\|_{L^2(\Gamma)}\,, & \operatorname{Im}\lambda\ge 0\,,\\ C\,\langle\lambda\rangle^{-\frac12}\,\log\langle\lambda^{-1}\rangle\,e^{-D_\Gamma \operatorname{Im}\lambda}\,\|f\|_{L^2(\Gamma)}\,, & \operatorname{Im}\lambda\le 0\,. \end{cases}$$

Proof. The following kernel bounds hold by (14), since $|x-y|$ is bounded above,

$$|G_0(\lambda,x,y)| \le C\,e^{-\operatorname{Im}\lambda|x-y|}\,\langle\lambda\rangle^{-\frac12}\,\log\langle\lambda^{-1}\rangle\,|x-y|^{-\frac12}\,.$$

By the Schur test and symmetry of the kernel, the operator norm of $G(\lambda)$ is bounded by the following

$$\sup_x \int_\Gamma |G_0(\lambda,x,y)|\,d\sigma(y)$$

where σ is 1-dimensional Hausdorff measure, which equals arclength measure on each Γ_j.

First consider $\operatorname{Im}\lambda\le 0$. Then $e^{-\operatorname{Im}\lambda|x-y|}\le e^{-D_\Gamma\operatorname{Im}\lambda}$ for $x,y\in\Gamma$. After rotation, we can write Γ_j as the graph $y_2=F_j(y_1)$ for y_1 in a compact set K_j, and with uniform Lipschitz bounds on F_j. Then on Γ_j we have $d\sigma(y)\approx dy_1$, and

$$\sup_x \int_{\Gamma_j} |x-y|^{-\frac12}\,d\sigma(y) \;\le\; C\,\sup_{x_1}\int_{K_j}|x_1-y_1|^{-\frac12}\,dy_1 \;\le\; C\,D_{K_j}^{1/2}\,.$$

For $\operatorname{Im}\lambda\ge 0$, we use instead the bound

$$\sup_{x_1}\int_{K_j} e^{-\operatorname{Im}\lambda|x_1-y_1|}\,|x_1-y_1|^{-\frac12}\,dy_1 \le C_j\,\langle\operatorname{Im}\lambda\rangle^{-\frac12}\,.$$

Summing over finitely many j then yields the desired bounds over Γ. □

4. Resolvent Bounds in the Upper Half Plane

In this section, we prove Theorems 1.2 and 1.3 for $\operatorname{Im}\lambda>0$. We assume that Γ is a finite union of compact subsets of embedded $C^{1,1}$ hypersurfaces, with $(d-1)$-dimensional Hausdorff measure. For $f\in L^2(\Gamma)$ we use $\gamma^*f=f\delta_\Gamma$ to denote the induced compactly supported distribution. If $|\lambda|\le 2$, the estimates of Theorems 1.2 and 1.3 follow from Theorem 3.1 in case $d=2$, and from the estimates (12) in case $d\ge 3$. So in this section we assume $|\lambda|\ge 2$.

For $\operatorname{Im}\lambda>0$ let $R_0(\lambda)=(-\Delta-\lambda^2)^{-1}$ be the operator with Fourier multiplier $(|\xi|^2-\lambda^2)^{-1}$. The estimates of both Theorem 1.2 and Theorem 1.3 are equivalent to bounds on the following quantity, for $f,g\in L^2(\Gamma)$,

$$Q_\lambda(f,g) := \int R_0(\lambda)(\gamma^*f)\,\overline{\gamma^*g}\,. \tag{15}$$

For $\operatorname{Im}\lambda>0$, the right hand side (15) agrees with the distributional pairing of $R_0(\lambda)(\gamma^*f)\in H^{\frac32-\epsilon}$ with $\gamma^*g\in H^{-\frac12-\epsilon}$, and hence by the Plancherel theorem

$$Q_\lambda(f,g) = \int \frac{\widehat{\gamma^*f}(\xi)\,\overline{\widehat{\gamma^*g}(\xi)}}{|\xi|^2-\lambda^2}\,d\xi\,. \tag{16}$$

We start with the following Lemma, which shows that bounds on $Q_\lambda(f,g)$ for λ in the upper half plane can be deduced from appropriate restriction bounds on the Fourier transform of $\gamma^* f$.

Lemma 4.1. *Suppose that for some $\alpha \in (0,1)$ the following estimate holds for $r > 0$,*

$$\int \left|\widehat{\gamma^* f}(\xi)\right|^2 \delta(|\xi| - r)\, d\xi \;\le C\,\langle r\rangle^\alpha \|f\|^2_{L^2(\Gamma)}\,. \tag{17}$$

Then, for λ in the upper half plane with $|\lambda| \ge 2$,

$$|Q_\lambda(f,g)| \le C\,|\lambda|^{\alpha-1}\log|\lambda|\,\|f\|_{L^2(\Gamma)}\|g\|_{L^2(\Gamma)}\,,$$

where Q_λ is as in (15)*.*

Proof. Consider first the integral in (16) over $\big||\xi| - |\lambda|\big| \ge 1$. Since $\big||\xi|^2 - \lambda^2\big| \ge \big||\xi|^2 - |\lambda|^2\big|$, by the Schwartz inequality and (17) this piece of the integral is bounded by

$$\|f\|_{L^2(\Gamma)}\|g\|_{L^2(\Gamma)} \int_{|r - |\lambda|| \ge 1} \langle r\rangle^\alpha \left| r^2 - |\lambda|^2 \right|^{-1} dr \le C\,|\lambda|^{\alpha-1}\log|\lambda|\,\|f\|_{L^2(\Gamma)}\|g\|_{L^2(\Gamma)}.$$

Next, if $\operatorname{Im}\lambda \ge 1$, then $\big||\xi|^2 - \lambda^2\big| \ge |\lambda|$, and by (17)

$$\left|\,\int_{||\xi| - |\lambda|| \le 1} \frac{\widehat{\gamma^* f}(\xi)\,\overline{\widehat{\gamma^* g}}(\xi)}{|\xi|^2 - \lambda^2}\, d\xi\,\right| \le C\,|\lambda|^{\alpha-1}\,\|f\|_{L^2(\Gamma)}\|g\|_{L^2(\Gamma)}.$$

Thus, we may restrict our attention to $0 \le \operatorname{Im}\lambda \le 1$ and $\big||\xi| - |\lambda|\big| \le 1$. For this piece we use that (17) implies

$$\int \left|\nabla_\xi\,\widehat{\gamma^* f}(\xi)\right|^2 \delta(|\xi| - r)\, d\xi \;\le C\,\langle r\rangle^\alpha \|f\|^2_{L^2(\Gamma)}\,, \tag{18}$$

due to the compact support of $\gamma^* f$.

We consider $\operatorname{Re}\lambda \ge 0$, the case $\operatorname{Re}\lambda \le 0$ following similarly, and write

$$\frac{1}{|\xi|^2 - \lambda^2} = \frac{1}{|\xi| + \lambda}\;\frac{\xi}{|\xi|}\cdot\nabla_\xi \log(|\xi| - \lambda)\,,$$

where the logarithm is well defined since $\operatorname{Im}(|\xi| - \lambda) < 0$. Let $\chi(r) = 1$ for $|r| \le 1$ and vanish for $|r| \ge \frac{3}{2}$. We then use integration by parts, together with (17) and (18) to bound

$$\left|\int \chi(|\xi| - |\lambda|)\,\frac{1}{|\xi| + \lambda}\,\widehat{\gamma^* f}(\xi)\,\overline{\widehat{\gamma^* g}}(\xi)\;\frac{\xi}{|\xi|}\cdot\nabla_\xi \log(|\xi| - \lambda)\, d\xi\,\right| \le C\,|\lambda|^{\alpha-1}\,\|f\|_{L^2(\Gamma)}\|g\|_{L^2(\Gamma)}.$$

□

The proof of Theorem 1.2 is thus accomplished by showing that (17) holds with $\alpha = \frac{1}{2}$ if Γ is a compact subset of a $C^{1,1}$ hypersurface, and with $\alpha = \frac{1}{3}$ if Γ is a compact subset of a strictly convex $C^{2,1}$ hypersurface. Since we work locally we assume that Γ is given by the graph $x_n = F(x')$, where by an extension argument we assume that F is a $C^{1,1}$ function (respectively $C^{2,1}$ function) defined on $\mathbb{R}^n$, and we replace surface measure on Γ by dx'. By scaling we may assume that $|\nabla F| \le \frac{1}{20}$, and by translation that $F(0) = 0$. Also, the estimate (17) is trivial by compactness of Γ if $r \le 1$, so we may assume $r \ge 1$.

The estimate (17) is equivalent, by duality, to an estimate in $L^2(\Gamma)$ on the restriction to Γ of eigenfunctions of the Laplacian. To see this, let $r \ge 1$, and let $\delta_{\mathbb{S}^{d-1}_r} = \delta(|\xi| - r)$ be surface measure on the sphere $\mathbb{S}_r^{d-1}$ of radius r. Assume that $g(\xi)$ is a function belonging to $L^2(\mathbb{S}_r^{d-1})$, and define

$$Tg(x) = \int e^{i\langle x,\xi\rangle} g(\xi)\,\delta(|\xi|-r)\,d\xi\,.$$

Let $\chi(x') \in C_c^\infty(\mathbb{R}^{d-1})$ be supported in the unit ball. By duality, (17) with $\alpha = \frac{1}{2}$ is then equivalent to the following estimate

$$\left(\int \bigl|(Tg)(x',F(x'))\bigr|^2 \chi(x')\,dx'\right)^{\frac{1}{2}} \leq C\,r^{\frac{1}{4}}\,\|g\|_{L^2(\mathbb{S}_r^{d-1})}\,, \tag{19}$$

and for $\alpha = \frac{1}{3}$ is equivalent to the same estimate with $r^{\frac{1}{4}}$ replaced by $r^{\frac{1}{6}}$.

The estimate (19) is known as a restriction estimate for eigenfunctions of the Laplacian. L^p generalizations in the setting of a smooth Riemannian manifold, with restriction to a smooth submanifold, were studied by Burq, Gérard and Tzvetkov in [6]. Semi-classical analogues were proved by Tacy [24] and Hassell-Tacy [15]. The L^2 estimates, again in the smooth setting, were noted by Tataru [26] as being a corollary of an estimate of Greenleaf and Seeger [14]. These estimates were generalized to the setting of restriction to smooth submanifolds in Riemannian manifolds with metrics of $C^{1,1}$ regularity by Blair [3]. In making a change of coordinates to flatten a submanifold the resulting metric has one lower order of regularity, thus the estimates of [3] do not apply directly to $C^{1,1}$ submanifolds, and so we include here the proof of the L^2 estimate on $C^{1,1}$ hypersurfaces of Euclidean space. The estimate with $\alpha = \frac{1}{3}$ for strictly convex $C^{2,1}$ hypersurfaces does follow from [3], so we consider here just the case of a general $C^{1,1}$ hypersurface and $\alpha = \frac{1}{2}$.

We in turn derive (19) from a square function estimate, Lemma 4.2. The estimate (20) is a characteristic trace estimate for solutions to the wave equation, but the proof more closely resembles that of dispersive estimates for the wave equation. Our proof of Lemma 4.2 is inspired by [3], although the analysis here is simpler since we work on Euclidean space, and seek only L^2 bounds on the restriction of eigenfunctions.

The reduction of (19) to the estimate (20) below is attained by letting $f = \psi Tg$, where $\psi \in C_c^\infty(\mathbb{R}^d)$ equals 1 on the ball of radius 3. Then $\cos(t\sqrt{-\Delta})f = \cos(tr)Tg$ for $|x| < 2$ and $|t| < 1$. On the other hand, $\hat{f} = \hat{\psi} * \bigl(g\,\delta_{\mathbb{S}_r^{d-1}}\bigr)$ is rapidly decreasing away from the sphere $|\xi| = r$, so the difference between $\hat{f}$ and its truncation to $\frac{3}{4}r \leq |\xi| \leq \frac{3}{2}r$ is easily handled. Also, a simple calculation shows that, uniformly over r,

$$\|\hat{\psi} * (g\,\delta_{\mathbb{S}_r^{d-1}})\|_{L^2(\mathbb{R}^d)} \leq C\,\|g\|_{L^2(\mathbb{S}_r^{d-1})}\,.$$

The proof of (19), hence of (17) and Theorem 1.2, is then completed by the following.

Lemma 4.2. *Suppose that $r \geq 1$ and $f \in L^2(\mathbb{R}^d)$, and $\hat{f}(\xi)$ is supported in the region $\frac{3}{4}r \leq |\xi| \leq \frac{3}{2}r$. If $F \in C^{1,1}(\mathbb{R}^{d-1})$ is real valued, with $\|\nabla F\|_{L^\infty} \leq \frac{1}{20}$, and $F(0) = 0$, then*

$$\left(\int_0^1 \Bigl\|\Bigl(\cos(t\sqrt{-\Delta})f\Bigr)(x',F(x'))\Bigr\|^4_{L^2(\mathbb{R}^{d-1},dx')}\,dt\right)^{\frac{1}{4}} \leq C\,r^{\frac{1}{4}}\|f\|_{L^2(\mathbb{R}^d)}\,. \tag{20}$$

Proof. Given a function F_r such that $\sup_{x'} |F_r(x') - F(x')| \leq r^{-1}$, then (20) holds if we can show that

$$\left(\int_0^1 \Bigl\|\Bigl(\cos(t\sqrt{-\Delta})f\Bigr)(x',F_r(x'))\Bigr\|^4_{L^2(\mathbb{R}^{d-1},dx')}\,dt\right)^{\frac{1}{4}} \leq C\,r^{\frac{1}{4}}\|f\|_{L^2(\mathbb{R}^d)}\,. \tag{21}$$

This follows from the fact that (21), together with the frequency localization of f and translation invariance, implies the gradient bound, uniformly over $s\in\mathbb{R}$,

$$\biggl(\int_0^1 \Bigl\|\partial_s\Bigl(\cos(t\sqrt{-\Delta})f\Bigr)(x',F_r(x')+s)\Bigr\|^4_{L^2(\mathbb{R}^{d-1},dx')}\,dt\biggr)^{\frac14}\le C\,r^{\frac54}\|f\|_{L^2(\mathbb{R}^d)}\,.$$

We will take F_r to be a mollification of the $C^{1,1}$ function F on the $r^{-\frac12}$ spatial scale. Precisely, let $F_r=\phi_{r^{1/2}}*F$, where $\phi_{r^{1/2}}=r^{\frac{d-1}{2}}\phi(r^{\frac12}x')$, with ϕ a Schwartz function on $\mathbb{R}^{d-1}$ of integral 1. Then

$$\sup_{x'}|F_r(x')-F(x')|\le C\,r^{-1}\,,\qquad \sup_{x'}|\nabla F_r(x')-\nabla F(x')|\le C\,r^{-\frac12}\,,$$

and F_r is a smooth function with derivative bounds

$$\sup_{x'}|\partial^\alpha_{x'}F_r(x')|\le C\,r^{\frac{|\alpha|-2}{2}}\,,\qquad |\alpha|\ge 2\,. \tag{22}$$

In establishing (21) we may replace $\cos(t\sqrt{-\Delta})$ by $\exp(it\sqrt{-\Delta})$, the bounds for $\exp(-it\sqrt{-\Delta})$ being similar. Let $Tf(t,x')=\bigl(\exp(it\sqrt{-\Delta})f\bigr)(x',F_r(x'))$. We deduce bounds for $T:L^2(\mathbb{R}^d)\to L^4([0,1],L^2(\mathbb{R}^{d-1}))$ from bounds for TT^*. Precisely, let $K_r(t-s,x-y)$ denote the kernel of the operator

$$\rho\bigl(r^{-1}D\bigr)\exp\bigl(i(t-s)\sqrt{-\Delta}\,\bigr)\,,\qquad D:=-i\partial\,,$$

where ρ is a smooth function supported in the region $\frac12<|\xi|<2$. It then suffices to show that

$$\biggl\|\int_0^1\!\!\int K_r\bigl(t-s,(x'-y',F_r(x')-F_r(y'))\bigr)\,f(s,y')\,dy'ds\biggr\|_{L^4([0,1],L^2(\mathbb{R}^{d-1}))}\le C\,r^{\frac12}\|f\|_{L^{4/3}([0,1],L^2(\mathbb{R}^{d-1}))} \tag{23}$$

since this implies $\|TT^*f\|_{L^4([0,1],L^2(\mathbb{R}^{d-1}))}\le C\,r^{\frac12}\|f\|_{L^{4/3}([0,1],L^2(\mathbb{R}^{d-1}))}$, and hence (21). We recall the Hardy-Littlewood-Sobolev inequality,

$$\||t|^{-\frac12}*f\|_{L^4(\mathbb{R})}\le C\|f\|_{L^{4/3}(\mathbb{R})}\,.$$

Translation invariance in t then shows that (23) is a consequence of the following fixed-time estimate, for $|t|<1$,

$$\biggl\|\int K_r\bigl(t,(x'-y',F_r(x')-F_r(y'))\bigr)\,f(y')\,dy'\biggr\|_{L^2(\mathbb{R}^{d-1})}\le C\,r^{\frac12}\,|t|^{-\frac12}\|f\|_{L^2(\mathbb{R}^{d-1})}\,. \tag{24}$$

If $|t|\le r^{-1}$, where we recall $r\ge 1$, then (24) follows by the Schur test, since if $|t|r\le 1$ then for any $N\ge 0$

$$|K_r(t,x-y)|\le C_N\,r^d\bigl(1+r\,|x-y|\bigr)^{-N}\,.$$

We thus restrict attention to $|t|>r^{-1}$, where we establish (24) using wave packet techniques that were developed to prove dispersive estimates for wave equations with $C^{1,1}$ coefficients; see [22].

To prove (24) for a given t with $|t|>r^{-1}$, we make an almost orthogonal decomposition $K_r=\sum_j K_j$ of the convolution kernel $K_r(t,\cdot)$. This decomposition is based on dividing the frequency space into essentially disjoint cubes of sidelength $\approx r^{\frac12}|t|^{-\frac12}$. On each of these cubes the phase of the wave operator is essentially linear in the frequency variable, and hence each term K_j behaves as a normalized convolution operator in x.

We fix t with $|t|\in[r^{-1},1]$, and let $\delta=r^{\frac12}|t|^{-\frac12}$. Let η_j count the elements of the lattice of spacing δ for which $|\eta_j|\in[\frac12 r,2r]$, and write

$$\rho(r^{-1}\xi)=\sum_j Q_j(\xi)\,,$$

where Q_j is supported in the cube of sidelength δ centered on η_j, and the following bounds hold on the derivatives of Q_j, uniformly over r, t and j,

$$\big|\partial_\xi^\alpha Q_j(\xi)\big|\le C_\alpha\,\delta^{-|\alpha|}\,. \tag{25}$$

We then write $K_r(t,x)=\sum K_j(x)$, where we suppress the dependence on r and t, and set

$$K_j(x)=(2\pi)^{-d}\int e^{i\langle x,\xi\rangle+it|\xi|}\,Q_j(\xi)\,d\xi\,.$$

The multiplier $t|\xi|-t|\eta_j|^{-1}\langle\eta_j,\xi\rangle$ satisfies the derivative bounds (25) on the support of Q_j, hence we may write

$$e^{i\langle x,\xi\rangle+it|\xi|}\,Q_j(\xi)=e^{i\langle x+t\,|\eta_j|^{-1}\eta_j,\xi\rangle}\,\tilde Q_j(\xi)\,,$$

with $\tilde Q_j$ having the same support and derivative conditions as Q_j. Consequently, we may write

$$K_j(x)=\delta^d\,e^{i\langle x,\eta_j\rangle+it|\eta_j|}\chi_j\big(\delta(x+t\,|\eta_j|^{-1}\eta_j)\big)\,,$$

where χ_j is a Schwartz function, with seminorm bounds independent of j. We let

$$\tilde K_j(x',y')=K_j\big(x'-y',F_r(x')-F_r(y')\big)\,.$$

It follows from the Schur test that

$$\|\tilde K_j\|_{L^2(\mathbb{R}^{d-1})\to L^2(\mathbb{R}^{d-1})}\le C\,\delta\,.$$

To handle the sum over j we establish the estimate

$$\|\tilde K_j\tilde K_i^*\|_{L^2\to L^2}+\|\tilde K_j^*\tilde K_i\|_{L^2\to L^2}\le C_N\,\delta^2\big(1+\delta^{-1}|\eta_i-\eta_j|\big)^{-N}\,, \tag{26}$$

from which the bound (24) follows by the Cotlar-Stein lemma. Since $\tilde K_j$ and $\tilde K_j^*$ have similar form, we restrict attention to the first term in (26).

The kernel $(\tilde K_j\tilde K_i^*)(x',z')$ has absolute value dominated by

$$\big|(\tilde K_j\tilde K_i^*)(x',z')\big|\le C\,\delta^{2d}\int\big(1+\delta\,|x+t\,|\eta_j|^{-1}\eta_j-y|\,\big)^{-N}\big(1+\delta\,|z+t\,|\eta_i|^{-1}\eta_i-y|\,\big)^{-N}\,dy' \tag{27}$$

where we use the notation $y=(y',F_r(y'))$, and similarly for x and z.

Suppose that $|(\eta_j)_n|\ge\frac14|\eta_j|$. Then since $|F_r(x')-F_r(y')|\le\frac1{10}|x'-y'|$,

$$\big|x'+t\,|\eta_j|^{-1}\eta_j'-y'\big|+10\big|F_r(x')+t\,|\eta_j|^{-1}(\eta_j)_n-F_r(y')\big|\ge 5\,t\,,$$

hence (27) and the Schur test leads to the bound

$$\|\tilde K_j\tilde K_i^*\|_{L^2\to L^2}\le C_N\,\delta^2\big(1+\delta\,t\big)^{-N}\,,$$

which is stronger than (26) since $|\eta_i-\eta_j|\le 6r$. The same estimate holds if $|(\eta_i)_n|\ge\frac14|\eta_i|$.

We thus assume that $|(\eta_j)_n|\le\frac14|\eta_j|$, and similarly for η_i. Consider then the case where $|(\eta_i-\eta_j)_n|\ge|(\eta_i-\eta_j)'|$. Then we have

$$\big|(|\eta_j|^{-1}\eta_j-|\eta_i|^{-1}\eta_i)_n\big|\ge\frac{1}{2+2\sqrt2}\,\big|(|\eta_j|^{-1}\eta_j-|\eta_i|^{-1}\eta_i)'\big|\,,$$

and since $\frac{1}{2}r \le |\eta_i|, |\eta_j| \le 2r$,

$$\big|(|\eta_j|^{-1}\eta_j - |\eta_i|^{-1}\eta_i)_n\big| \ge \frac{1}{4\sqrt{2}}\, r^{-1}|\eta_i - \eta_j|\,.$$

Then since $|\nabla F_r| \le \frac{1}{10}$,

$$|x'-z'+t(|\eta_j|^{-1}\eta_j-|\eta_i|^{-1}\eta_i)'|+10\,|F_r(x')-F_r(z')+t(|\eta_j|^{-1}\eta_j-|\eta_i|^{-1}\eta_i)_n| \ge \frac{5}{4\sqrt{2}}\,\delta^{-2}|\eta_j-\eta_i|\,,$$

hence (27) and the Schur test show that $\|\tilde{K}_j\tilde{K}_i^*\|_{L^2\to L^2} \le C_N\,\delta^2\big(1+\delta^{-1}|\eta_j-\eta_i|\big)^{-N}$ as desired.

We thus consider the case that $|(\eta_j-\eta_i)_n| \le |(\eta_j-\eta_i)'|$. In this case we need use the oscillations of the kernels to bound $\|\tilde{K}_j\tilde{K}_i^*\|_{L^2\to L^2}$. Up to a factor of modulus 1, the kernel $(K_jK_i^*)(x',z')$ can be written as

$$\delta^{2d}\int e^{-i\langle y',\eta_j'-\eta_i'\rangle - iF_r(y')(\eta_j-\eta_i)_n}\,\chi_j\big(\delta(x+t\,|\eta_j|^{-1}\eta_j-y)\big)\,\overline{\chi_i}\big(\delta(z+t\,|\eta_i|^{-1}\eta_i-y)\big)\,dy'\,,$$

where again $y=(y',F_r(y'))$, and similarly for x and z. Since $|\nabla F_r(y')| \le \frac{1}{10}$, and $|(\eta_j-\eta_i)_n| \le |\eta_j'-\eta_i'|$, we have

$$|\eta_j'-\eta_i'+\nabla F_r(y')(\eta_j-\eta_i)_n| \ge \tfrac{1}{2}|\eta_j-\eta_i|\,.$$

Using the estimates (22), and that $r^{\frac{1}{2}} \le \delta$, an integration by parts argument dominates the kernel $(K_jK_i^*)(x',z')$ by

$$\delta^{2d}\big(1+\delta^{-1}|\eta_j-\eta_i|\big)^{-N}\int\big(1+\delta\,|x+t\,|\eta_j|^{-1}\eta_j-y|\,\big)^{-N}\big(1+\delta\,|z+t\,|\eta_i|^{-1}\eta_i-y|\,\big)^{-N}\,dy'\,,$$

which leads to the desired norm bounds, concluding the proof of (26), and hence of Lemma 4.2. □

We conclude this section with the proof of Theorem 1.3. The estimates for $0<\operatorname{Im}\lambda<1$ follow from Theorem 1.2, so we consider $\operatorname{Im}\lambda \ge 1$. As above, we need to establish bounds on $Q_\lambda(f,g)$ defined by (16).

First consider the case that $f=g$ and Γ is a graph $x_n=F(x')$. We then have uniform bounds

$$\sup_{\xi_n}\int\big|\widehat{\gamma^*f}(\xi',\xi_n)\big|^2\,d\xi' \le C\,\|f\|^2_{L^2(\Gamma)}\,.$$

We use the lower bound $\big||\xi|^2-\lambda^2\big| \ge |\lambda|\,|\operatorname{Im}\lambda|$ to dominate

$$\int_{|\xi_n|\le 2|\lambda|}\frac{\big|\widehat{\gamma^*f}(\xi)\big|^2}{\big||\xi|^2-\lambda^2\big|}\,d\xi \le C\,\langle\operatorname{Im}\lambda\rangle^{-1}\,\|f\|^2_{L^2(\Gamma)}\,.$$

For $|\xi_n| \ge 2|\lambda|$ we have $\big||\xi|^2-\lambda^2\big| \gtrsim |\xi_n|^2$, hence

$$\int_{|\xi_n|\ge 2|\lambda|}\frac{\big|\widehat{\gamma^*f}(\xi)\big|^2}{\big||\xi|^2-\lambda^2\big|}\,d\xi \le C\,\langle\lambda\rangle^{-1}\,\|f\|^2_{L^2(\Gamma)}\,.$$

The case $f \ne g$ and Γ a finite union of graphs then follows by a partition of unity argument and the Schwarz inequality. □

5. Resolvent Bounds in the Lower Half Plane

We consider $d \geq 2$, and Γ as in Theorem 1.2. For $\lambda \in \mathbb{R}$ and $\lambda \neq 0$, the resolvent $R_0(\lambda)$ is defined as the limit $R_0(\lambda + i0)$ from $\operatorname{Im}\lambda > 0$. The estimates of Theorem 1.2 for $\operatorname{Im}\lambda > 0$, proved in Section 4, show that, for $\lambda \in \mathbb{R}$ with $|\lambda| > 2$, and for some $a > 0$ and $b \in \{0, 1\}$, we have

$$\|\gamma\, R_0(\lambda)\, \gamma^*\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\, |\lambda|^{-a} \left(\log|\lambda|\right)^b .$$

In this section we extend this to bounds for $\operatorname{Im}\lambda < 0$, to complete the proof of Theorem 1.2.

Lemma 5.1. *Suppose that Γ is as in Theorem 1.2 and that for $\lambda \in \mathbb{R}$, $|\lambda| \geq 2$, the following holds*

$$\|\gamma\, R_0(\lambda)\, \gamma^*\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\, |\lambda|^{-a} (\log|\lambda|)^b .$$

Then for $\operatorname{Im}\lambda \leq 0$, $|\lambda| \geq 2$, and $\arg\lambda \in [-\pi, 0] \cup [\pi, 2\pi]$ in the case that d is even,

$$\|\gamma\, R_0(\lambda)\, \gamma^*\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\, |\lambda|^{-a} (\log|\lambda|)^b e^{-D_\Gamma \operatorname{Im}\lambda}$$

where D_Γ is the diameter of Γ.

Proof. First consider the case that d is odd. Suppose that $\|f\|_{L^2(\Gamma)} = \|g\|_{L^2(\Gamma)} = 1$, and consider the function

$$F(\lambda) = e^{-iD_\Gamma \lambda}\, \lambda^a \left(\log\lambda\right)^{-b} Q_\lambda(f, g)\,, \qquad \operatorname{Im}\lambda \leq 0\,,\ \ |\lambda| \geq 2\,,$$

where $\log\lambda$ is defined for $\arg\lambda \in (\frac{\pi}{2}, \frac{5\pi}{2})$. Then $|F(\lambda)| \leq C$ for $\lambda \in \mathbb{R} \setminus [-2, 2]$ and for $|\lambda| = 2$. On the other hand, the resolvent kernel bounds (12) and the Schur test show that $|F(\lambda)|$ has at most polynomial growth in λ for $\operatorname{Im}\lambda \leq 0$, since the kernel $|x - x'|^{2-d}$ is integrable over a $d-1$ dimensional hypersurface. It follows by the Phragmén-Lindelöf theorem that $|F(\lambda)| \leq C$ in the lower half plane.

In the case that d is even, we note that the bounds of the lemma hold for $R_0(\lambda)$ if $\arg\lambda = 2\pi$ and $|\lambda| \geq 2$. This follows since $R_0(e^{i\pi}\lambda) - R_0(\lambda)$ satisfies the same bounds as $R_0(\lambda)$ for $\arg\lambda = 0$, and by (13) we have $R_0(e^{2i\pi}\lambda) - R_0(e^{i\pi}\lambda) = R_0(e^{i\pi}\lambda) - R_0(\lambda)$. We may thus apply the Phragmén-Lindelöf theorem on the sheet $\pi \leq \arg\lambda \leq 2\pi$. A similar argument works for $-\pi \leq \arg\lambda \leq 0$. □

6. Application to Resonance Free Regions

In this section we establish Theorem 1.1. First, we demonstrate the meromorphic continuation of $R_V(\lambda)$ from $\operatorname{Im}\lambda \gg 1$ to $\lambda \in \mathbb{C}$ (to the logarithmic cover in even dimensions) following arguments similar to those in the case where $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$. We assume Γ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces. We use ρ to denote a function in $C_c^\infty(\mathbb{R}^d)$ with $\rho = 1$ on a neighborhood of Γ; the following results hold for any such choice of ρ. For λ in the domain of $R_0(\lambda)$ we define

$$K(\lambda) = (V \otimes \delta_\Gamma) R_0(\lambda) = \gamma^* V \gamma R_0(\lambda)\,.$$

The operator $K(\lambda)\rho : H^{-1}(\mathbb{R}^d) \to H^{-\frac{1}{2}-\epsilon}_{\mathrm{comp}}$ is compact on $H^{-1}(\mathbb{R}^d)$ by Rellich's embedding theorem. Furthermore, $I + K(\lambda)\rho$ is invertible if $\operatorname{Im}\lambda \gg 1$. To see this, note that $g + K(\lambda)\rho g = 0$ and $g \in H^{-1}(\mathbb{R}^d)$ implies that $g = \gamma^* f$ where $f \in L^2(\Gamma)$. It follows that $f + VG(\lambda) f = 0$, which implies $f = 0$ for $\operatorname{Im}\lambda \gg 1$ by Theorem 1.3. This also shows that $I + K(\lambda)\rho$ is invertible on $H^{-1}(\mathbb{R}^d)$ if and only if $I + VG(\lambda)$ is invertible on $L^2(\Gamma)$.

Then $(I+K(\lambda)\rho)^{-1}$ is a meromorphic family of Fredholm operators on $H^{-1}(\mathbb{R}^d)$ for λ in the domain of $R_0(\lambda)$. This follows by analytic Fredholm theory, see e.g. Proposition 7.4 of [27, Chapter 9]. Note that for $d=1$ the domain is $\mathbb{C}\backslash\{0\}$. We prove meromorphicity at 0 for $d=1$ following Proposition 6.2 below; for now if $d=1$ we assume $\lambda\in\mathbb{C}\backslash\{0\}$.

Since $K=\gamma^*V\gamma R_0$, we have that

$$(I+K(\lambda)\rho)^{-1}\gamma^*=\gamma^*(I+VG(\lambda))^{-1}\,, \tag{28}$$

where $(I+VG(\lambda))^{-1}$ acts on $L^2(\Gamma)$. Hence,

$$\begin{aligned}(I+K(\lambda)\rho)^{-1}&=I-(I+K(\lambda)\rho)^{-1}K(\lambda)\rho\\&=I-\gamma^*(I+VG(\lambda))^{-1}V\gamma R_0(\lambda)\rho\,.\end{aligned}$$

The meromorphic extension of the resolvent $R_V(\lambda)$ for $-\Delta_{V,\Gamma}$ then equals, for any ρ as above,

$$\begin{aligned}R_V(\lambda)&=R_0(\lambda)\left(I+K(\lambda)\rho\right)^{-1}\left(I-K(\lambda)(1-\rho)\right)\\&=\Big(R_0(\lambda)-R_0(\lambda)\gamma^*(I+VG(\lambda))^{-1}V\gamma R_0(\lambda)\rho\Big)\left(I-K(\lambda)(1-\rho)\right).\end{aligned} \tag{29}$$

In particular, given $g\in H^{-1}_{\mathrm{comp}}$ we can take $\rho g=g$ to obtain

$$R_V(\lambda)g=R_0(\lambda)g-R_0(\lambda)\gamma^*(I+VG(\lambda))^{-1}V\gamma R_0(\lambda)g\,. \tag{30}$$

Consequently, $R_V(\lambda):H^{-1}_{\mathrm{comp}}\to H^1_{\mathrm{loc}}$, and its image is λ-outgoing.

The resolvent set Λ is defined as the set of poles of $R_V(\lambda)$. Since

$$\big(I-K(\lambda)(1-\rho)\big)\big(I+K(\lambda)(1-\rho)\big)=I\,,$$

the preceding arguments show that Λ agrees with the poles of $(I+VG(\lambda))^{-1}$, except possibly $\lambda=0$ when $d=1$. If $\|G(\lambda)\|_{L^2\to L^2}<\|V\|^{-1}_{L^2\to L^2}$, then $I+VG(\lambda)$ is invertible by Neumann series. By Theorem 1.2 and (2), when $\operatorname{Im}\lambda<0$ this is the case provided that $|\lambda|>2$ and

$$|\operatorname{Im}\lambda|\le D_\Gamma^{-1}\big(a\log|\lambda|-\log C-\log(\log|\lambda|)\big)$$

for some C, where $a=\frac12$ or $\frac23$ or 1 accordingly, which completes the proof of Theorem 1.1. □

Remark: For λ in the domain of $R_0(\lambda)$, the $L^2(\Gamma)$ kernel of $I+G(\lambda)V$ is in one-to-one correspondence with the kernel of $I+VG(\lambda)$ by the map $h\to Vh$. That $(I+G(\lambda)V)h=0$ implies $(I+VG(\lambda))Vh=0$ is immediate. Conversely, $(I+VG(\lambda))f=0$ expresses $f=-VG(\lambda)f:=Vh$, and $(I+G(\lambda)V)h=-G(\lambda)(I+VG(\lambda))f=0$.

To equate resonances to λ-outgoing solutions of (7), we use the following extension of the Rellich uniqueness theorem.

Proposition 6.1 (Rellich uniqueness)**.** *If λ belongs to the domain of $R_0(\lambda)$, then a global λ-outgoing solution to $(-\Delta-\lambda^2)u=0$ must vanish identically.*

Proof. For $0<\arg\lambda<\pi$ and g a compactly supported distribution, $R_0(\lambda)g$ is exponentially decreasing in $|x|$, so Green's identities yield, for $u=R_0(\lambda)g$ and for $R\gg1$, that

$$u(x)=\int_{|y|=R}\Big(G_0(\lambda,x,y)\,\partial_{\nu'}u(y)-\partial_{\nu'_y}G_0(\lambda,x,y)u(y)\Big)\,d\sigma(y)\,,\qquad |x|>R\,.$$

By analytic continuation this holds for all λ in the domain of $R_0(\lambda)$. If u is an entire solution then the right hand side is real-analytic in R, and we may let $R \to 0$ to deduce that $u \equiv 0$. $\square$

Proposition 6.2. *For λ in the domain of $R_0(\lambda)$, there is a one-to-one correspondence of λ-outgoing solutions $u \in H^1_{\mathrm{loc}}$ to (7) and solutions $f \in L^2(\Gamma)$ to $(I + VG(\lambda))f = 0$, given by $u = R_0(\lambda)(\gamma^* f)$, and $f = -V\gamma u$.*

Proof. If $(I + VG(\lambda))f = 0$, $f \in L^2(\Gamma)$, then $u = R_0(\lambda)(\gamma^* f)$ is a λ-outgoing solution to $-\Delta_{V,\Gamma} u = \lambda^2 u$. Indeed $u \in H^1_{\mathrm{loc}}$ and is λ-outgoing by definition, $(-\Delta - \lambda^2)u = \gamma^* f$, and $(V \otimes \delta_\Gamma)u = \gamma^* V G(\lambda) f = -\gamma^* f$.

Conversely, if $u \in H^1_{\mathrm{loc}}$ is a λ-outgoing solution to $-\Delta u - \lambda^2 u = -(V \otimes \delta_\Gamma)u$, then by Proposition 6.1

$$u = -R_0(\lambda)(V \otimes \delta_\Gamma)u = -\int_\Gamma G_0(\lambda, x, y)\,(V\gamma u)(y)\,. \tag{31}$$

Hence if $f = -V\gamma u$, then $f + VG(\lambda)f = 0$. By (31) the correspondence between u and $V\gamma u$ is one-to-one. As a result, the space of solutions u for given λ is finite dimensional, since it is in one-to-one correspondence with the kernel of a Fredholm operator. $\square$

The case $d = 1$ and $\lambda = 0$. For $d = 1$, we need to prove that $R_V(\lambda)$ is meromorphic at $\lambda = 0$, and equate existence of 0-outgoing (i.e. separately constant near $\pm\infty$) solutions of $-\Delta_{V,\Gamma} u = 0$ to 0 being a pole. When $d = 1$, $I + VG(\lambda)$ is a matrix valued meromorphic function for $\lambda \in \mathbb{C}$, invertible on $L^2(\Gamma) \equiv \mathbb{C}^m$ for $\operatorname{Im}\lambda \gg 1$, so $\det(I + VG(\lambda))$, hence $(I + VG(\lambda))^{-1}$, is meromorphic on $\mathbb{C}$. Equation (30), which holds for $\lambda \in \mathbb{C}\backslash\{0\}$, and meromorphicity of $R_0(\lambda)$ on $\mathbb{C}$, then establishes meromorphicity of $R_V(\lambda)$ on $\mathbb{C}$, in particular that 0 is either a regular point or a pole. Furthermore, since γ^* has finite dimensional range, so do the singular terms of $R_V(\lambda)$ at $\lambda = 0$. It remains to show that $R_V(\lambda)$ is singular at 0 if and only if there is a nontrivial solution $u \in H^1_{\mathrm{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ to $\Delta u = \gamma^* V\gamma u$, since $u \in L^\infty(\mathbb{R})$ is equivalent to 0-outgoing for such u. By the discussion preceding (36) below, a pole at 0 implies existence of a 0-outgoing solution to (7). Conversely, if $R_V(\lambda)$ is holomorphic at $\lambda = 0$, then (28) and the identity (see [11, Section 2.2])

$$(I + K(\lambda)\rho)^{-1} = I - \gamma^* V\gamma R_V(\lambda)\rho$$

shows that the matrix $(I + VG(\lambda))^{-1}$ is then holomorphic at $\lambda = 0$. Considering adjoints, we must then have

$$\|f\|_{L^2(\Gamma)} \le C\,\|(I + G(\lambda)V)f\|_{L^2(\Gamma)}\,, \qquad f \in L^2(\Gamma)\,, \quad |\lambda| \ll 1\,. \tag{32}$$

Suppose $u \in H^1_{\mathrm{loc}} \cap L^\infty$ satisfies $\Delta u = \gamma^* V\gamma u$. Let $\Gamma = \{x_1, \dots, x_m\} \subset \mathbb{R}$, and $V\gamma u = (c_1, \dots, c_m) \in \mathbb{C}^m$. Then

$$u(x) = \frac{1}{2}\sum_{x_j \in \Gamma} c_j\,|x - x_j|\, + ax + b\,, \quad \text{for some } \ a, b \in \mathbb{C}\,.$$

Since $u \in L^\infty$ we must have $\sum_j c_j = 0$ and $a = 0$. Hence, with $E_{ij} = -\frac{1}{2}|x_i - x_j|$, we have

$$\langle \gamma 1, V\gamma u\rangle = 0\,, \qquad (I + EV)\gamma u = \gamma b\,,$$

where 1 and b are constant functions on $\mathbb{R}$. Since $G(\lambda)_{jk} = -(2i\lambda)^{-1}\exp(i\lambda|x_j - x_k|)$, then for $f \in L^2(\Gamma)$

$$(I + G(\lambda)V)f = -(2i\lambda)^{-1}\langle \gamma 1, Vf\rangle\,\gamma 1 + (I + EV)f + \mathcal{O}(\lambda)\,f\,.$$

Assume first that $V\gamma 1 \neq 0$, and take $f = \gamma u + 2i\lambda \, \|V\gamma 1\|^{-2} \, V\gamma b$. Then $(I + G(\lambda)V)f = \mathcal{O}(\lambda)$, contradicting (32) unless $\gamma u = 0$, hence $u \equiv 0$.

We conclude by showing that $R_V(\lambda)$ regular at $\lambda = 0$ implies $V\gamma 1 \neq 0$. To see this, note that if $V\gamma 1 = 0$ (in which case $-\Delta_{V,\Gamma} 1 = 0$ would give a 0-outgoing solution) then $K(\lambda)$ is regular at $\lambda = 0$, since for $g \in H^{-1}_{\mathrm{comp}}$

$$V\gamma R_0(\lambda)g = V\gamma\Big(R_0(\lambda)g + (2i\lambda)^{-1}\langle 1, g\rangle\, 1\Big) = -V\gamma \int \left(\frac{e^{i\lambda|x-y|} - 1}{2i\lambda}\right) g(y)\, dy\,.$$

Then (29) shows that for $g \in H^{-1}_{\mathrm{comp}}$, by taking $\rho = 1$ on a neighborhood of $\mathrm{supp}(g) \cup \Gamma$, we can write $R_0(\lambda)g = R_V(\lambda)(I + K(\lambda))g$, hence $R_V(\lambda)$ must be singular at 0 since $R_0(\lambda)$ is. □

Proof of Proposition 1.6. Suppose now that $\Gamma = \partial\Omega$ for a compact domain $\Omega \subset \mathbb{R}^d$ with $C^{1,1}$ boundary. Assume also that $V : H^{\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$. Then the analysis leading to (10) shows that a λ-outgoing solution of (7) with $u \in H^1_{\mathrm{loc}}$ belongs to $\mathcal{E}_{2,\mathrm{loc}}$ and satisfies the transmission problem (8). Conversely, suppose $u \in \mathcal{E}_{2,\mathrm{loc}}$ satisfies (8). For $w \in C_c^\infty(\mathbb{R}^d)$, Green's identities yield

$$\int_{\mathbb{R}^d} u\,(-\Delta - \lambda^2)w = \int_{\partial\Omega} (\partial_\nu u + \partial_{\nu'} u)\,\gamma w = -\int_{\partial\Omega} (V\gamma u)\,\gamma w\,.$$

Hence u is a λ-outgoing H^1_{loc} distributional solution to $(-\Delta - \lambda^2)u + (V \otimes \delta_{\partial\Omega})u = 0$, and by the above λ is a resonance. □

7. Resonance Expansion for the Wave Equation

In this section we prove Theorems 1.4 and 1.5. We will use the following representation of the wave group $U(t)$ acting on $g \in L^2_{\mathrm{comp}}(\mathbb{R}^d)$,

$$U(t)g = \frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} e^{-it\lambda} R_V(\lambda) g\, d\lambda\,, \tag{33}$$

where $\alpha \geq 1$ is chosen so that $\mu_j < \alpha$ for all j, where $-\mu_j^2$ are the negative eigenvalues of $-\Delta_{V,\Gamma}$ with $\mu_j > 0$. This representation follows by the spectral theorem and the resolvent estimates we establish in this section; see (43). The expansion (6) is proven by a contour integration argument applied to (33). We start this section by studying the structure of the resolvent $R_V(\lambda)$ near its poles, and then prove norm estimates on $R_V(\lambda)$ that justify the change of contour used to prove Theorem 1.4. We then establish higher order estimates on $R_V(\lambda)$, which are used to prove Theorem 1.5.

Let Λ denote the set of resonances; since we work in odd dimensions Λ is a discrete subset of $\mathbb{C}$. The elements of Λ such that $\operatorname{Im}\lambda > 0$ consist of $i\mu_j$ where $-\mu_j^2$ are the eigenvalues of $-\Delta_{V,\Gamma}$ in $(-\infty, 0)$ with $\mu_j > 0$. That there are only a finite number of such eigenvalues follows by relative compactness of $V \otimes \delta_\Gamma$ with respect to $-\Delta$. The resolvent near $i\mu_j$ takes the form

$$R_V(\lambda) \;=\; \frac{-\Pi_{\mu_j}}{\lambda^2 + \mu_j^2} + \text{holomorphic} \;=\; \frac{i\,\Pi_{\mu_j}}{2\mu_j(\lambda - i\mu_j)} + \text{holomorphic}\,,$$

where Π_{μ_j} is projection onto the $-\mu_j^2$-eigenspace of $-\Delta_{V,\Gamma}$. In particular we note that

$$\mathrm{Res}\big(e^{-it\lambda} R_V(\lambda), i\mu_j\big) = i(2\mu_j)^{-1} e^{t\mu_j}\, \Pi_{\mu_j}\,. \tag{34}$$

We note that if there is a compactly supported eigenfunction u for $-\mu_j^2$, then $-i\mu_j$ must also be a resonance. To see this, by compact support of u we can write

$$u(x) = \int G_0(-i\mu_j, x, y)(-\Delta + \mu_j^2)u(y)\, dy = -R_0(-i\mu_j)(V \otimes \delta_\Gamma)u\,,$$

hence u is also $-i\mu_j$ outgoing, and $-i\mu_j$ is a resonance by the results of Section 6.

In contrast to the case of $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, there may be resonances $\lambda \in \mathbb{R} \setminus \{0\}$. For an example in one dimension of V and Γ with a positive (hence embedded) eigenvalue, consider $\Gamma = \left\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\right\}$, and V given by

$$(V\gamma u)(x) = \begin{cases} u(0)\,, & x = \pm\frac{\pi}{2} \\ u(\frac{\pi}{2}) + u(-\frac{\pi}{2})\,, & x = 0 \end{cases}$$

Then the function

$$u(x) = \begin{cases} \cos(x)\,, & |x| \le \frac{\pi}{2} \\ 0\,, & |x| \ge \frac{\pi}{2} \end{cases}$$

is compactly supported, and satisfies $-\Delta_{V,\Gamma}u - u = 0$. It is λ-outgoing for both $\lambda = \pm 1$ by the argument above, hence yields resonances at $\lambda = \pm 1$. Using piecewise linear functions one can also produce an example of a compactly supported eigenfunction with eigenvalue 0, and using piecewise combinations of $\{e^x, e^{-x}\}$ produce a compactly supported eigenfunction with eigenvalue -1, for appropriate choices of V and Γ.

For $\lambda \in \mathbb{R} \setminus \{0\}$ and any dimension d, a λ-outgoing solution $u \in H^1_{\mathrm{loc}}$ to $-\Delta_{V,\Gamma}u = \lambda^2 u$ must in fact be a compactly supported eigenfunction. To see this, observe that for $R \gg 1$

$$0 = \int_{|x|\le R} \overline{u}\left(-\Delta u + (V \otimes \delta_\Gamma)u - \lambda^2 u\right) = \int_{|x|\le R} \left(|\nabla u|^2 - \lambda^2|u|^2\right) + \int_{|x|=R} \overline{u}\,\partial_\nu u + \int_\Gamma \overline{\gamma u}\, V\gamma u$$

shows that $\mathrm{Im}\int_{|x|=R} \overline{u}\,\partial_\nu u = 0$. The proof of Proposition 1.1 and Lemma 1.2 of [27, Chapter 9] then show that $u \equiv 0$ on $|x| \ge R_0$, hence by analytic continuation u vanishes on the unbounded component of $\mathbb{R}^d \setminus \Gamma$. For $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d)$, unique continuation (see [21, Theorem XIII.63]) would yield $u \equiv 0$. For singular potentials and non-local V unique continuation can fail by the example above, but we note that if Γ coincides with the boundary of the unbounded component of $\mathbb{R}^d \setminus \Gamma$ then there are no resonances $\lambda \in \mathbb{R} \setminus \{0\}$, since in that case $\gamma u = 0$, hence $(V \otimes \delta_\Gamma)u = 0$. Thus u is a compactly supported eigenfunction of $-\Delta$ on $\mathbb{R}^d$, and must vanish identically.

The resonances in $\mathbb{R} \setminus \{0\}$ form a finite set by Theorem 1.1. By Proposition 6.2 and the preceding, $\lambda \in \mathbb{R} \setminus \{0\}$ is a resonance if and only if λ^2 is an eigenvalue of $-\Delta_{V,\Gamma}$, and the real resonances are thus symmetric about 0. We indicate them by $\pm\nu_j$, with $\nu_j > 0$. The spectral bound $\|R_V(\lambda)\|_{L^2\to L^2} \le C\,\epsilon^{-1}|\,\mathrm{Im}\,\lambda|^{-1}$, for $|\,\mathrm{Re}\,\lambda| \ge \epsilon$ and $\mathrm{Im}\,\lambda > 0$, shows that the pole at ν_j is simple. By inspection, for $\mathrm{Im}\,\lambda > 0$ near $\pm\nu_j$ we have

$$R_V(\lambda) \;=\; \frac{-\Pi_{\nu_j}}{\lambda^2 - \nu_j^2} + \text{holomorphic} \;=\; \frac{\mp\Pi_{\nu_j}}{2\nu_j(\lambda \mp \nu_j)} + \text{holomorphic}\,,$$

where Π_{ν_j} is projection onto the ν_j^2 eigenspace, hence

$$\mathrm{Res}\big(e^{-it\lambda}R_V(\lambda), \pm\nu_j\big) = \mp(2\nu_j)^{-1}e^{\mp it\nu_j}\,\Pi_{\nu_j}\,. \tag{35}$$

The nature of the residue at 0 depends on the dimension d. For $d \ge 5$, λ-outgoing solutions to (7) for $\lambda = 0$ must be square-integrable, hence if $0 \in \Lambda$ there is a corresponding eigenfunction. For $d = 1$, a square-integrable solution to (7) must be compactly supported; there may also be 0-outgoing solutions (i.e. constant near $\pm\infty$) that are not eigenfunctions for 0. For $d = 3$, if $0 \in \Lambda$ there may be square-integrable and/or non square-integrable solutions to (7) since, depending on whether the integral over Γ of $f = V\gamma u$ vanishes or not, $u = R_0(0)\gamma^* f$ satisfies $|u| \lesssim |x|^{-2}$ or $|u| \approx |x|^{-1}$ for $|x| \gg 1$.

For $|\lambda| \ll 1$ and $\operatorname{Im}\lambda > 0$, the spectral bound $\|R_V(\lambda)\|_{L^2\to L^2} \le C(|\lambda| \operatorname{Im}\lambda)^{-1}$ shows that

$$R_V(\lambda) = -\frac{\Pi_0}{\lambda^2} + \frac{i\mathcal{P}_0}{\lambda} + \text{holomorphic}\,.$$

Since $R_V^*(-\overline{\lambda}) = R_V(\lambda)$ for $\operatorname{Im}\lambda > 0$, it follows that Π_0 and $\mathcal{P}_0$ are symmetric maps of L^2_{comp} to $\mathcal{D}_{loc}$, in that $\langle \mathcal{P}_0 g, h\rangle = \langle g, \mathcal{P}_0 h\rangle$ for $g, h \in L^2_{\mathrm{comp}}$, similarly for Π_0, and their images are solutions of (7) with $\lambda = 0$. Since Π_0 is bounded on $L^2(\mathbb{R}^d)$ it is then self-adjoint, and since it is the identity on the 0-eigenspace we see that Π_0 is projection onto the 0-eigenspace of $-\Delta_{V,\Gamma}$. For $d \ge 3$ the range of Π_0 is 0-outgoing, since $u = -R_0(0)(V \otimes \delta_\Gamma)u$ when $u \in \mathcal{D}$ solves $-\Delta_{V,\Gamma} u = 0$. We remark that the arguments of [11, Section 3.3] show that $\mathcal{P}_0 = 0$ for $d \ge 5$, and that the range of $\mathcal{P}_0$ is 0-outgoing if $d = 3$, although we do not use that here. To see that the range of Π_0 and $\mathcal{P}_0$ are 0-outgoing when $d = 1$, we note that $(\partial_x - i\operatorname{sgn}(x)\lambda)(R_V(\lambda)g)(x) = 0$ for $|x| \gg 1$ and $g \in L^2_{\mathrm{comp}}$. The range of Π_0 is supported in the convex hull of Γ when $d = 1$ (hence is 0-outoing), and by letting $\lambda \to 0$ this implies $\partial_x(\mathcal{P}_0 g)(x) = 0$ for $|x| \gg 1$, hence the range of $\mathcal{P}_0$ is 0-outgoing.

We can then write

$$\operatorname{Res}\big(e^{-it\lambda} R_V(\lambda), 0\big) = it\,\Pi_0 + i\mathcal{P}_0\,. \tag{36}$$

The remaining resonances form a discrete set $\{\lambda_k\} \subset \{\operatorname{Im}\lambda < 0\}$, with respective order $m_R(\lambda_k)$. Since $\lambda_k \ne 0$, the Laurent expansion of $R_V(\lambda)$ about λ_k can be written in the following form

$$R_V(\lambda) \;=\; i \sum_{\ell=1}^{m_R(\lambda_k)} \frac{(-\Delta_{V,\Gamma} - \lambda_k^2)^{\ell-1}\mathcal{P}_{\lambda_k}}{(\lambda^2 - \lambda_k^2)^\ell} + \text{holomorphic}\,.$$

Here $\mathcal{P}_{\lambda_k} : L^2_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$ is given by

$$\mathcal{P}_{\lambda_k} = -\frac{1}{2\pi} \oint_{\lambda_k} R_V(\lambda)\, 2\lambda\, d\lambda\,,$$

and $(-\Delta_{V,\Gamma} - \lambda_k^2)^{m_R(\lambda_k)}\mathcal{P}_{\lambda_k} = 0$. We can thus write

$$\operatorname{Res}\big(e^{-it\lambda} R_V(\lambda), \lambda_k\big) \;=\; i \sum_{\ell=0}^{m_R(\lambda_k)-1} t^\ell\, e^{-it\lambda_k}\, \mathcal{P}_{\lambda_k,\ell} \tag{37}$$

where $\mathcal{P}_{\lambda_k,\ell} : L^2_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}$. When $\ell = m_R(\lambda_k) - 1$, $\mathcal{P}_{\lambda_k,\ell} g$ is λ_k-outgoing, as seen by writing the Laurent expansion of $R_V(\lambda)$ in terms of that for $(I + K(\lambda)\rho)^{-1}$. In particular, if $m_R(\lambda_k) = 1$, then $\operatorname{Res}\big(e^{-it\lambda} R_V(\lambda), \lambda_k\big) = i(2\lambda_k)^{-1} e^{-it\lambda_k}\,\mathcal{P}_{\lambda_k}$, where $\mathcal{P}_{\lambda_k}$ maps L^2_{comp} to λ_k-outgoing solutions of $(-\Delta_{V,\Gamma} - \lambda_k^2)u = 0$.

7.1. **Resolvent Estimates.** We first establish bounds on the cutoff of $R_V(\lambda)$, for λ in the resonance free region established in Section 6.

Lemma 7.1. *Suppose that Γ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces. Then for all $\epsilon > 0$ there exists $R < \infty$, so that if $\chi \in C_c^\infty(\mathbb{R}^d)$ equals 1 on a neighborhood of Γ, $|\operatorname{Re}\lambda| > R$, and $\operatorname{Im}\lambda \geq -(\frac{1}{2}D_\Gamma^{-1} - \epsilon)\log(|\operatorname{Re}\lambda|)$, then*

$$\begin{aligned}
\|\chi R_V(\lambda)\chi g\|_{L^2} &\leq C\,\langle\lambda\rangle^{-1}\,e^{2D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{L^2}\,,\\
\|\chi R_V(\lambda)\chi g\|_{H^1} &\leq C\,e^{2D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{L^2}\,,\\
\|\chi R_V(\lambda)\chi g\|_{\mathcal{D}} &\leq C\,\langle\lambda\rangle\,e^{2D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{L^2}\,,
\end{aligned}$$

where $R_V(\lambda)$ is the meromorphic continuation of $(-\Delta_{V,\Gamma} - \lambda^2)^{-1}$, $D_\chi = \operatorname{diam}(\operatorname{supp}\chi)$, and $(\operatorname{Im}\lambda)_- = \max(0, -\operatorname{Im}\lambda)$. If $\operatorname{Im}\lambda \geq 1$, $|\operatorname{Re}\lambda| > R$, then the estimates hold with $\chi \equiv 1$, setting $D_\chi(\operatorname{Im}\lambda)_- = 0$.

Remark: The region in which this estimate is valid can be improved by replacing $\frac{1}{2}$ by $\frac{2}{3}$ if the components of Γ are subsets of strictly convex $C^{2,1}$ hypersurfaces.

Proof. We recall the Sobolev estimates for the cutoff of the free resolvent if $|\lambda| \geq 1$, see e.g. [11, Chapter 3]

$$\|\chi R_0(\lambda)\chi\|_{H^s\to H^t} \leq C\langle\lambda\rangle^{t-s-1}e^{D_\chi(\operatorname{Im}\lambda)_-}\,,\qquad s\leq t\leq s+2\,.$$

In addition, when $\operatorname{Im}\lambda \geq 1$ these estimates hold globally, that is with $\chi \equiv 1$ and taking $D_\chi(\operatorname{Im}\lambda)_- = 0$.

This in turn leads to the following restriction estimates

$$\begin{aligned}
\|\gamma R_0(\lambda)\chi g\|_{L^2(\Gamma)} &\leq C\,\langle\lambda\rangle^{-s-\frac{1}{2}}e^{D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{H^s}\,,\quad -\tfrac{3}{2} < s < \tfrac{1}{2}\,,\\
\|\gamma\nabla R_0(\lambda)\chi g\|_{L^2(\Gamma)} &\leq C\,\langle\lambda\rangle^{-s+\frac{1}{2}}e^{D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{H^s}\,,\quad -\tfrac{1}{2} < s < \tfrac{3}{2}\,.
\end{aligned}\tag{38}$$

To prove (38) we apply the following trace bound separately on each component of Γ,

$$\|\gamma g\|_{L^2(\Gamma)} \leq C_{t,t'}\,\|g\|_{H^t}^{\theta}\,\|g\|_{H^{t'}}^{1-\theta}\,,\qquad 0\leq t<\tfrac{1}{2}<t'\,,\quad \theta(t-\tfrac{1}{2})+(1-\theta)(t'-\tfrac{1}{2})=0\,.\tag{39}$$

The estimate (39) follows by considering the case of a graph $x_n = F(x')$, and applying Hölder's inequality in x' and the following scale-invariant one dimensional estimate in x_n

$$\|g\|_{L^\infty(\mathbb{R})} \leq C_{t,t'}\,\|\,|D|^t g\|_{L^2(\mathbb{R})}^{\theta}\,\|\,|D|^{t'} g\|_{L^2(\mathbb{R})}^{1-\theta}\,,$$

with t, t', θ as in (39). This estimate follows by fixing r so that $\|\,|D|^t g_r\|_{L^2(\mathbb{R})} = \|\,|D|^{t'} g_r\|_{L^2(\mathbb{R})}$, where $g_r(x) = g(rx)$, and noting $\|\widehat{g_r}\|_{L^1(\mathbb{R})} \leq \frac{1}{2}C_{t,t'}\big(\|\,|\xi|^t\widehat{g_r}\|_{L^2(\mathbb{R})} + \|\,|\xi|^{t'}\widehat{g_r}\|_{L^2(\mathbb{R})}\big)$ if $t < \frac{1}{2} < t'$, for some $C_{t,t'} < \infty$.

By duality (38) implies the following extension estimate,

$$\|\chi R_0(\lambda)\gamma^* f\|_{H^s} \leq C\,\langle\lambda\rangle^{s-\frac{1}{2}}\,e^{D_\chi(\operatorname{Im}\lambda)_-}\,\|f\|_{L^2(\Gamma)}\,,\quad -\tfrac{1}{2} < s < \tfrac{3}{2}\,.\tag{40}$$

Now fix $g \in L^2(\mathbb{R}^d)$, and set $u = R_V(\lambda)\chi g$. Then by (30) we have $u = R_0(\lambda)\chi g - w$, where

$$w = R_0(\lambda)\gamma^*(I + VG(\lambda))^{-1}V\gamma R_0(\lambda)\chi g\,.$$

By Theorem 1.2, for $|\operatorname{Re}\lambda|$ large enough and $\operatorname{Im}\lambda \geq -(\frac{1}{2}D_\Omega^{-1} - \epsilon)\log(|\operatorname{Re}\lambda|)$, the operator $I + VG(\lambda)$ is invertible on $L^2(\Gamma)$, and we have

$$\|(I+VG(\lambda))^{-1}\|_{L^2(\Gamma)\to L^2(\Gamma)} \leq C\,,\qquad \|VG(\lambda)\|_{L^2(\Gamma)\to L^2(\Gamma)} < 1\,.$$

It follows from (38) that, for $-\frac{3}{2} < s < \frac{1}{2}$,

$$\|(I+VG(\lambda))^{-1}V\gamma R_0(\lambda)\chi g\|_{L^2(\Gamma)} \le C\,\langle\lambda\rangle^{-s-\frac{1}{2}}\,e^{D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{H^s}\,.$$

Then (40) gives the following, for $-\frac{3}{2} < s < \frac{1}{2}$, and with global bounds if $\operatorname{Im}\lambda \ge 1$,

$$\|\chi w\|_{L^2} \le C\,\langle\lambda\rangle^{-s-1}e^{2D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{H^s}\,, \tag{41}$$

$$\|\chi w\|_{H^1} \le C\,\langle\lambda\rangle^{-s}\,e^{2D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{H^s}\,. \tag{42}$$

By the $L^2 \to H^t$ bounds for $\chi R_0(\lambda)\chi$ the same holds for $s=0$ with w replaced by u, which yields the bounds of Lemma 7.1 except for the ones on $\|\chi u\|_{\mathcal{D}}$.

To obtain bounds on $\|\chi u\|_{\mathcal{D}}$, we write

$$\Delta(\chi u) = -\chi^2 g + 2(\nabla\chi)\cdot\nabla u + (\Delta\chi)u - \lambda^2\chi u + (V\otimes\delta_\Gamma)u\,,$$

and note by (41) and (42) that

$$\|(\nabla\chi)\cdot\nabla u\|_{L^2} + \|(\Delta\chi)u\|_{L^2} + \langle\lambda\rangle^2\|\chi u\|_{L^2} \le C\,\langle\lambda\rangle\,e^{2D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{L^2}\,.$$

Consequently,

$$\|\Delta_{V,\Gamma}(\chi u)\|_{L^2} \le C\,\langle\lambda\rangle\,e^{2D_\chi(\operatorname{Im}\lambda)_-}\|g\|_{L^2}\,,$$

yielding the desired bound on $\|\chi u\|_{\mathcal{D}}$. □

7.2. **Proof of Theorem 1.4.** We prove here the case $N=1$ of Theorem 1.4; that is, that the expansion holds with bounds on $\|\chi E_A(t)\chi\|_{L^2\to\mathcal{D}}$. The case $N \ge 2$ will be handled following the proof of Theorem 1.5. We follow the treatment in [25] and suppose that $g \in H^s$ for some $0 < s < \frac{1}{2}$, then proceed by density of H^s in L^2. As above write

$$R_V(\lambda)\chi g = w(\lambda) + R_0(\lambda)\chi g\,.$$

Choose $\alpha \ge 1$ so that $\mu_j < \alpha$ for all j, where $-\mu_j^2$ are the negative eigenvalues of $-\Delta_{V,\Gamma}$. By the spectral theorem we can write

$$\begin{aligned} U(t)\chi g &= \frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} e^{-it\lambda}R_V(\lambda)\chi g\,d\lambda \\ &= \frac{1}{2\pi}\int_{-\infty+i\alpha}^{\infty+i\alpha} e^{-it\lambda}\big(w(\lambda)+R_0(\lambda)\chi g\big)\,d\lambda\,. \end{aligned} \tag{43}$$

The integral is norm convergent in $L^2(\mathbb{R}^d)$, by (41) and the norm convergence of the free resolvent integral. After localizing by χ on the left, for t sufficiently large we seek to deform the contour $\mathbb{R}+i\alpha$ to

$$\Sigma_A = \big\{\lambda\in\mathbb{C} : \operatorname{Im}\lambda = -A - c\log\big(2+|\operatorname{Re}\lambda|\big)\big\}$$

where we choose $c < \frac{1}{2}D_\Gamma^{-1}$, and assume A is such that there are no resonances on Σ_A. We will show that the integral over Σ_A is norm convergent for $g\in H^s$ if $s>0$, so to justify the contour change we need to show that for t sufficiently large the integrals over

$$\gamma_{\pm R}(v) = \big\{\pm R + iv \,:\, -\big(A + c\log(2+R)\big) \le v \le \alpha\big\}\,,\quad\text{and}\quad \gamma_{R,\infty} = \{x+i\alpha \,:\, |x|\ge R\}$$

tend to 0 as $R\to\infty$. Note that for R large enough, Theorem 1.1 shows that there are no resonances between $\mathbb{R}+i\alpha$ and Σ_A with $|\operatorname{Re}\lambda|\ge R$, and hence none on $\gamma_{\pm R}$.

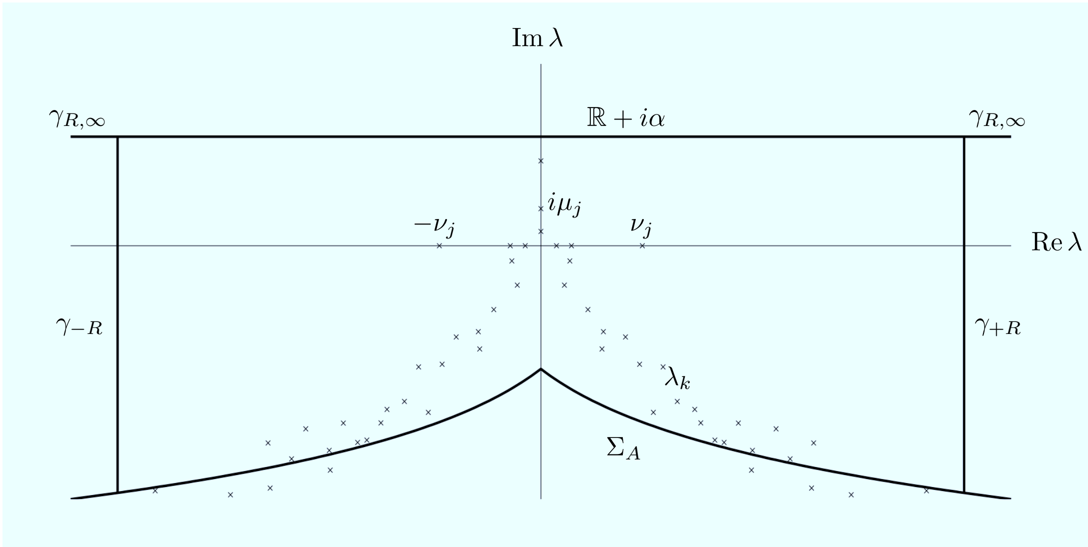

FIGURE 2. The various contours used in Section 7.2 to obtain the resonance expansion in odd dimensions.

We introduce the following notation,

$$E_\gamma(t)g = \frac{1}{2\pi}\int_\gamma e^{-it\lambda} R_V(\lambda) g \, d\lambda \, .$$

Then for $t > 2D_\chi$, and R large enough,

$$\|\chi E_{\gamma_{\pm R}}(t)\chi g\|_{L^2} \le C\, e^{\alpha t}\langle R\rangle^{-1}\big(\alpha + A + c\log(2+R)\big)\|g\|_{L^2} \to 0 \quad \text{as} \quad R \to \infty \, .$$

The norm convergence of (43) shows that $\|\chi E_{\gamma_{R,\infty}}\chi g\|_{L^2} \to 0$ as $R \to \infty$. We then assume $c(t - 2D_\chi) \ge 3$ and calculate

$$\|\chi E_{\Sigma_A}(t)\chi g\|_{\mathcal{D}} \le C_{A,\chi}\, e^{-A(t-2D_\chi)} \int_{-\infty}^{\infty} e^{-3\log(2+|R|)}\langle A + |R|\,\rangle \, dR \le C_{A,\chi}\, e^{-At}\, \|g\|_{L^2} \, .$$

In particular the integral is norm convergent, and the contour deformation is allowed.

Thus, if we let Ω_A denote the collection of poles of $R_V(\lambda)$ in the set $\operatorname{Im}\lambda > -A - c\log\big(2 + |\operatorname{Re}\lambda|\big)$, then

$$\chi U(t)\chi g = \chi E_{\Sigma_A}(t)\chi g - i\chi \sum_{z\in\Omega_A} \operatorname{Res}\big(e^{-it\lambda}R_V(\lambda), z\big)\chi g \, ,$$

and by density this holds for $g \in L^2(\mathbb{R}^d)$. Observe that if $g \in L^2_{\mathrm{comp}}$ then we can take $\chi = 1$ on the support of g, and drop the cutoff χ to write a global equality in L^2_{loc}. To have estimates on the remainder in $\mathcal{D}$, though, requires cutting off by χ and taking $t > 2D_\chi + C$, which is required for $\chi U(t)\chi$ to map L^2 into $\mathcal{D} \subset H^1$. The expressions (34), (36), (35), and (37) now complete the proof of Theorem 1.4 for $N = 1$, where we observe that the terms from poles in Ω_A with $\operatorname{Im}\lambda \le -A$ can be absorbed into $E_A(t)$. □

7.3. **Higher Order Estimates for Smooth Domains.** We start with the following lemma, where we now assume that $\Gamma = \partial\Omega$ is C^∞, and that $V : H^s(\partial\Omega) \to H^s(\partial\Omega)$

for all $s \geq 0$. Recall that we set $\mathcal{E}_0 = L^2(\mathbb{R}^d)$, and for $N \geq 1$,

$$\mathcal{E}_N = H^1(\mathbb{R}^d) \cap \big(H^N(\Omega) \oplus H^N(\mathbb{R}^d \setminus \overline{\Omega}) \big) .$$

In this setting $\mathcal{D}$ equals the subspace of $\mathcal{E}_2$ satisfying $\partial_\nu u + \partial_{\nu'} u + V \gamma u = 0$.

Lemma 7.2. *Suppose that $\partial\Omega$ is of regularity C^∞, and $N \geq 0$. Then for all $\epsilon > 0$ there exists $R < \infty$, so that if $|\mathrm{Re}\,\lambda| > R$, $|\,\mathrm{Im}\,\lambda| \leq (\frac{1}{2} D_\Omega^{-1} - \epsilon) \log(|\mathrm{Re}\,\lambda|)$, and $\chi \in C_c^\infty(\mathbb{R}^d)$ equals 1 on a neighborhood of $\overline{\Omega}$, then*

$$\big\| \chi \big(R_V(\lambda) - R_V(-\lambda) \big) \chi g \big\|_{\mathcal{E}_N} \leq C_N \, \langle \lambda \rangle^{N-1} \, e^{2D_\chi |\,\mathrm{Im}\,\lambda|} \, \|g\|_{L^2} .$$

Proof. We proceed by induction on N. By Lemma 7.1, the result holds for $N = 0, 1, 2$. We assume then that the result is true for integers less than or equal to N.

Letting $u = \big(R_V(\lambda) - R_V(-\lambda) \big) \chi g$, we write

$$\Delta(\chi u) = 2(\nabla \chi) \cdot \nabla u + (\Delta \chi) u - \lambda^2 \chi u + (V \otimes \delta_{\partial\Omega}) u .$$

By the induction hypothesis,

$$\|(\Delta\chi)u\|_{H^{N-1}(\Omega) \oplus H^{N-1}(\mathbb{R}^d \setminus \overline{\Omega})} + \|\chi u\|_{H^{N-1}(\Omega) \oplus H^{N-1}(\mathbb{R}^d \setminus \overline{\Omega})} \leq C \, \langle \lambda \rangle^{N-2} \, e^{2D_\chi |\,\mathrm{Im}\,\lambda|} \, \|g\|_{L^2} ,$$

$$\|(\nabla\chi) \cdot \nabla u\|_{H^{N-1}(\Omega) \oplus H^{N-1}(\mathbb{R}^d \setminus \overline{\Omega})} + \|V \gamma u\|_{H^{N-\frac{1}{2}}(\partial\Omega)} \leq C \, \langle \lambda \rangle^{N-1} \, e^{2D_\chi |\,\mathrm{Im}\,\lambda|} \, \|g\|_{L^2} .$$

Proposition 8.1 then gives the desired result for $\mathcal{E}_{N+1}$. □

We now present the proof of Theorem 1.5. We use the notation from the proof of Theorem 1.4 above. We first note that

$$\frac{1}{2\pi} \int_{\Sigma_A} e^{-it\lambda} R_V(-\lambda) \, d\lambda \; = \; - \sum_{\mu_j > A + \log 2} (2\mu_j)^{-1} e^{-t\mu_j} \Pi_{\mu_j} ,$$

where the completion of the contour to the lower half plane is justified by Lemma 7.1 and the rapid decrease of $e^{-it\lambda}$ for $t > 0$. We thus can write

$$\chi E_{\Sigma_A}(t) \chi g = \frac{1}{2\pi} \int_{\Sigma_A} e^{-it\lambda} \chi \big(R_V(\lambda) - R_V(-\lambda) \big) \chi g \, d\lambda \; - \sum_{\mu_j > A + \log 2} (2\mu_j)^{-1} e^{-t\mu_j} \chi \Pi_{\mu_j} \chi g .$$

Assume $c(t - 2D_\chi) \geq N + 1$, by Lemma 7.2 the norm in $\mathcal{E}_N$ of the integral term is dominated by

$$C_{A,\chi} \, e^{-A(t - 2D_\chi)} \int_{-\infty}^{\infty} e^{-(N+1)\log(2+|R|)} \langle A + |R| \, \rangle^{N-1} \, dR \leq C_{A,\chi,N} \, e^{-At} \, \|g\|_{L^2} .$$

It remains to show that for the eigenvalues $-\mu_j^2$ with $\mu_j > A$, and the resonances λ_k with $\mathrm{Im}\,\lambda_k < -A$, then

$$e^{-t\mu_j} \|\chi \Pi_{\mu_j} \chi g\|_{\mathcal{E}_N} + \|\chi \mathrm{Res}\big(e^{-it\lambda} R_V(\lambda), \lambda_k \big) \chi g\|_{\mathcal{E}_N} \leq C_{A,\chi,N} \, e^{-tA} \, \|g\|_{L^2} ,$$

since the difference of $\chi E_A(t) \chi$ and $\chi E_{\Sigma_A}(t) \chi$ is a sum of such terms.

A similar argument to the proof of Lemma 7.2 gives the bound

$$\|\Pi_{\mu_j} g\|_{\mathcal{E}_N} \leq C_N \, \langle \mu_j \rangle^N \|g\|_{L^2} ,$$

which handles the eigenvalues. To handle the resonances in the lower half plane, consider first the case that $-\lambda_k$ is not a pole (that is, $\lambda_k \neq -i\mu_j$ for any j). We can then write

$$\mathrm{Res}\big(e^{-it\lambda} R_V(\lambda), \lambda_k \big) = \frac{1}{2\pi i} \oint_{\lambda_k} e^{-it\lambda} \big(R_V(\lambda) - R_V(-\lambda) \big) \, d\lambda ,$$

and the estimate follows from Lemma 7.2, by choosing a small contour about λ_k which is contained in $\operatorname{Im}\lambda < -A$. In the case that $-\lambda_k$ is a pole, hence an eigenvalue, then the term $R_V(-\lambda)$ contributes an eigenvalue projection, which is handled as above. □

We now complete the proof of Theorem 1.4 by considering the case $N \geq 2$. Eigenfunctions clearly belong to $\mathcal{D}_N$, and by an induction argument we have $\|\chi\Pi_{\mu_j}\chi g\|_{\mathcal{D}_N} \leq C_N\langle\mu_j\rangle^{2N}\|g\|_{L^2}$. The proof then follows from that of Theorem 1.5, using the following lemma.

Lemma 7.3. *Suppose that Γ is a finite union of compact subsets of $C^{1,1}$ hypersurfaces, and $N \geq 1$. Then for all $\epsilon > 0$ there exists $R < \infty$ so that if $|\operatorname{Re}\lambda| > R$, $|\operatorname{Im}\lambda| \leq (\frac{1}{2}D_\Gamma^{-1} - \epsilon)\log(|\operatorname{Re}\lambda|)$, and $\chi \in C_c^\infty(\mathbb{R}^d)$ equals 1 on a neighborhood of Γ, then*

$$\big\|\chi\big(R_V(\lambda) - R_V(-\lambda)\big)\chi g\big\|_{\mathcal{D}_N} \leq C\,\langle\lambda\rangle^{2N-1}\,e^{2D_\chi|\operatorname{Im}\lambda|}\,\|g\|_{L^2}\,.$$

Proof. The result was proven in Lemma 7.1 for $N = 1$. We then proceed by induction, writing

$$\begin{aligned}\Delta_{V,\Gamma}\,\chi\big(R_V(\lambda) - R_V(-\lambda)\big)\chi g &= \Big(\big[\Delta,\chi\big] - \lambda^2\chi\Big)\big(R_V(\lambda) - R_V(-\lambda)\big)\chi g\\ &= \Big(2\nabla\chi\cdot\nabla + (\Delta\chi) - \lambda^2\chi\Big)\big(R_V(\lambda) - R_V(-\lambda)\big)\chi g\,.\end{aligned}$$

By induction, and since $\operatorname{supp}(\Delta\chi) \subset \operatorname{supp}(\chi)$,

$$\|\big((\Delta\chi) - \lambda^2\chi\big)\big(R_V(\lambda) - R_V(-\lambda)\big)\chi g\|_{\mathcal{D}_{N-1}} \leq C\,\langle\lambda\rangle^{2N-1}\,e^{2D_\chi|\operatorname{Im}\lambda|}\,\|g\|_{L^2}\,. \tag{44}$$

By Lemma 7.1, if $\chi_1 \in C_c^\infty$ with $\operatorname{supp}(\chi_1) \subset \operatorname{supp}(\chi)$, and $u = \big(R_V(\lambda) - R_V(-\lambda)\big)\chi g$,

$$\langle\lambda\rangle\,\|\chi_1 u\|_{L^2} + \|\chi_1 u\|_{H^1} \leq C\,e^{2D_\chi|\operatorname{Im}\lambda|}\,\|g\|_{L^2}\,.$$

On the complement of Γ, the function $u = \big(R_V(\lambda) - R_V(-\lambda)\big)\chi g$ satisfies $-\Delta u = \lambda^2 u$. Since $\nabla\chi$ vanishes on a neighborhood of Γ, and $\operatorname{supp}(\nabla\chi) \subset \operatorname{supp}(\chi)$, an induction argument and elliptic regularity yields

$$\|\nabla\chi\cdot\nabla\big(R_V(\lambda) - R_V(-\lambda)\big)\chi g\|_{H^{2N-1}} \leq C\,\langle\lambda\rangle^{2N-1}e^{2D_\chi|\operatorname{Im}\lambda|}\,\|g\|_{L^2}\,,\quad N \geq 1\,.$$

Since $H^{2N-1}_{\mathrm{comp}}(\mathbb{R}^d\setminus\Gamma) \subset \mathcal{D}_{N-1}$ with continuous inclusion, this term also satisfies the bound of (44), and the result follows. □

8. The Transmission Property for $C^{1,1}$ Domains

We provide here a proof of the transmission estimate, Proposition 8.2, that we used in Section 2 to establish H^2 regularity of solutions on the complement of $\partial\Omega$ in case $\partial\Omega$ is of $C^{1,1}$ regularity. In case $\partial\Omega$ is smooth, the following estimate, which we used in the proof of Lemma 7.2, is well known; see [5], and in particular Theorems 9 and 10 of [12]. We record it here for reference.

Proposition 8.1. *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $\partial\Omega$ is locally the graph of a C^∞ function. Let $G_0(x,y)$ be Green's kernel for Δ^{-1}, and define the single layer potential map by*

$$S\ell f(x) = \int_{\partial\Omega} G_0(x,y)\,f(y)\,d\sigma(y)\,.$$

Then for $\chi \in C_c^\infty(\mathbb{R}^d)$, and $N \geq -1$, $\chi S\ell$ is a continuous map from $H^{N+\frac{1}{2}}(\partial\Omega) \to H^{N+2}(\Omega) \oplus H^{N+2}(\mathbb{R}^d \setminus \overline{\Omega})$.

Additionally, for $N \geq 0$ the map

$$\big(\chi G_0 \chi g\big)(x) = \chi(x) \int G_0(x,y)\,\chi(y)\,g(y)\,dy$$

is a continuous map from $H^N(\Omega) \oplus H^N(\mathbb{R}^d \setminus \overline{\Omega})$ to $H^{N+2}(\Omega) \oplus H^{N+2}(\mathbb{R}^d \setminus \overline{\Omega})$.

We need the same result for $N = 0$ and $\partial\Omega$ of $C^{1,1}$ regularity, in which case just the single layer potential result is nontrivial.

Proposition 8.2. *Suppose that $\Omega \subset \mathbb{R}^d$ is a bounded open set, and $\partial\Omega$ is locally the graph of a $C^{1,1}$ function. Let $G_0(x,y)$ be Green's kernel for Δ^{-1}, and let*

$$S\ell f(x) = \int_{\partial\Omega} G_0(x,y)\,f(y)\,d\sigma(y)\,.$$

Then for $\chi \in C_c^\infty(\mathbb{R}^d)$, $\chi S\ell$ is a continuous map from $H^{\frac{1}{2}}(\partial\Omega) \to H^2(\Omega) \oplus H^2(\mathbb{R}^d \setminus \overline{\Omega})$.

Proof. Since the kernel is smooth away from the diagonal we may work locally, and assume that $\partial\Omega$ is given as a graph $x_n = F(x')$, with $F \in C^{1,1}(\mathbb{R}^{d-1})$. Since surface measure $d\sigma(y) = m(y')\,dy'$ where m is Lipschitz, we can absorb the m into f. Assuming then that $f \in C_c^1(\mathbb{R}^{d-1})$, consider the maps

$$T'f(x) = (\nabla_{x'} S\ell f)(x', F(x') + x_d) \;=\; c_d \int \frac{(x'-y')\,f(y')\,dy'}{\big(|x'-y'|^2 + |x_d + F(x') - F(y')|^2\big)^{\frac{d}{2}}}$$

$$T_d f(x) = (\partial_{x_d} S\ell f)(x', F(x') + x_d) \;=\; c_d \int \frac{(x_d + F(x') - F(y'))\,f(y')\,dy'}{\big(|x'-y'|^2 + |x_d + F(x') - F(y')|^2\big)^{\frac{d}{2}}}$$

We seek $H^{\frac{1}{2}} \to H^1(x_d \neq 0)$ bounds for both terms. We have $\partial_{x_d} T' = \nabla_{x'} T_d - (\nabla_{x'} F)\partial_{x_d} T_d$, and since $\Delta S\ell f = 0$, for $x_d \neq 0$ we can write

$$(1 + |\nabla_{x'} F|^2)\partial_{x_d} T_d f = \nabla_{x'} T' f - (\nabla_{x'} F)\nabla_{x'} T_d f\,.$$

Thus it suffices to prove $H^{\frac{1}{2}} \to L^2$ bounds for $\chi \nabla_x' T'$ and $\chi \nabla_{x'} T_d$.

By the dual of the trace estimate we have

$$\|\chi S\ell f\|_{H^1} \leq C\,\|f\|_{H^{-1/2}(\partial\Omega)}\,,$$

and hence we can bound

$$\|\chi\, T'(\nabla_{y'} f)\|_{L^2} + \|\chi\, T_d(\nabla_{y'} f)\|_{L^2} \leq C\,\|f\|_{H^{1/2}(\partial\Omega)}\,.$$

The desired bound will thus follow from showing that

$$\big\|\chi\big[\nabla_{x'}, T'\big] f\big\|_{L^2} + \big\|\chi\big[\nabla_{x'}, T_d\big] f\big\|_{L^2} \leq C\,\|f\|_{L^2(\partial\Omega)}\,. \tag{45}$$

One can write $\big(\chi\big[\nabla_{x'}, T_d\big] f\big)(x) = \int K(x', x_d, y')\,f(y')\,dy'$, where

$$K(x', x_d, y') = \big(\nabla_{x'} + \nabla_{y'}\big)\frac{(x_d + F(x') - F(y'))}{\big(|x'-y'|^2 + |x_d + F(x') - F(y')|^2\big)^{\frac{d}{2}}}\,,$$

and one verifies that $|K(x', x_d, y')| \lesssim \big(x_d^2 + |x'-y'|^2\big)^{(1-d)/2}$ since ∇F is Lipschitz. Consequently,

$$\sup_{x'} \int_{|y'|\le L} |K(x', x_d, y')|\, dy' + \sup_{y'} \int_{|x'|\le L} |K(x', x_d, y')|\, dx' \le C_L \, \log\langle x_d^{-1}\rangle \,.$$

The bound (45) for this term is obtained by applying the Schur test in x' for each x_d, followed by integration over x_d, where we fix L so f and χ are supported in $|x'| \le L$. The corresponding kernel of $\chi\big[\nabla_{x'}, T'\big]$ satisfies the same bounds, which completes the proof of Proposition 8.2. □

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