An Introduction to Curvelets

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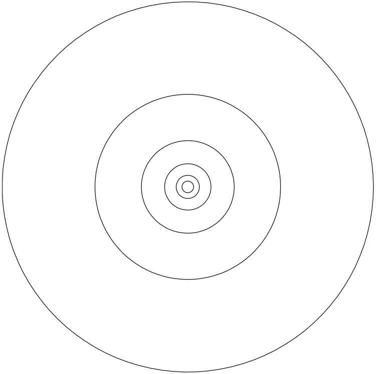
Curvelets

- A curvelet frame $\{\varphi_\gamma\}$ is a wave packet frame on $L^2(\mathbb{R}^2)$ based on second dyadic decomposition.

$$
\begin{aligned}
f(x) &= \sum_\gamma c_\gamma \, \varphi_\gamma(x) \\
c_\gamma &= \int f(x) \, \overline{\varphi_\gamma(x)} \, dx
\end{aligned}
$$

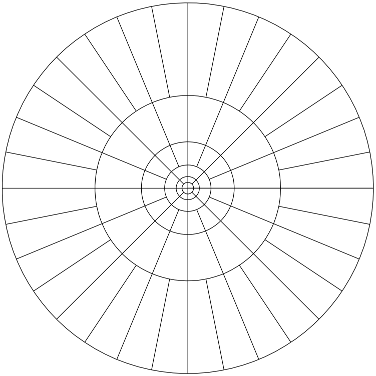
Dyadic Decomposition

Frequency shells: $2^k < |\xi| < 2^{k+1}$



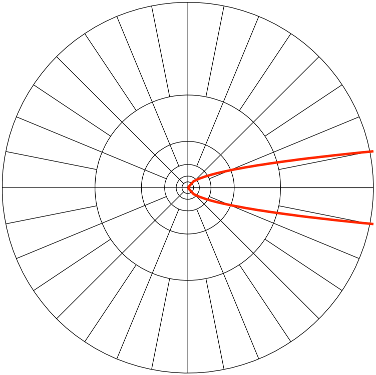
Second Dyadic Decomposition

Angular Sectors: $\angle(\omega,\xi) \leq 2^{-k/2}$



Second Dyadic Decomposition

Parabolic scaling: $\Delta\xi_2 \sim \sqrt{\xi_1}$



Second Dyadic Decomposition

Associated partition of unity:

$$1 = \hat{\psi}_0(\xi)^2 + \sum_{k=0}^{\infty} \sum_{\omega=1}^{2^{k/2}} \hat{\psi}_{\omega,k}(\xi)^2$$

$$\mathrm{supp}(\hat{\psi}_{\omega,k}) \subset \left\{ \xi : |\xi| \approx 2^k \,,\ \left|\omega - \tfrac{\xi}{|\xi|}\right| \lesssim 2^{-k/2} \right\}$$

Second dyadic decomposition of f:

$$1 = \hat{\psi}_0(\xi)^2 \, \hat{f}(\xi) + \sum_{k=0}^{\infty} \sum_{\omega=1}^{2^{k/2}} \hat{\psi}_{\omega,k}(\xi)^2 \, \hat{f}(\xi)$$

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Final step: expand $\hat{\psi}_{\omega,k}(\xi)\,\hat{f}(\xi)$ in Fourier series

If $\mathrm{supp}(g(\xi)) \subset L_1 \times L_2$ rectangle:

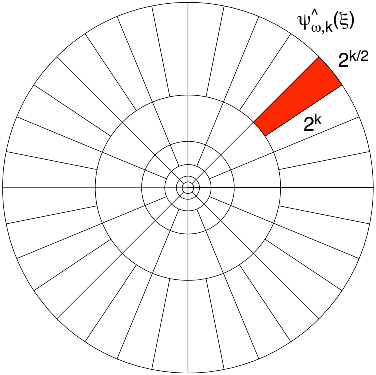
$$g(\xi) = (L_1 L_2)^{-1/2} \sum_{p,q} c_{p,q} e^{-i\,p\,\xi_1 - i\,q\,\xi_2}$$

$$c_{p,q} = (L_1 L_2)^{-1/2} \int g(\xi) e^{i\,p\,\xi_1 + i\,q\,\xi_2}$$

Points (p, q) belong to dilated lattice:

$$p, q \in \frac{2\pi}{L_1}\mathbb{Z} \times \frac{2\pi}{L_2}\mathbb{Z}$$

$\hat{\psi}_{\omega,k}(\xi)\hat{f}(\xi)$ supported in rotated $2^k \times 2^{k/2}$ rectangle



Points (p, q) belong to rotated $2^{-k} \times 2^{-k/2}$ lattice $\Xi_{\omega,k}$

Let $(p, q) = x \in \Xi_{\omega,k}$

$$c_{x,\omega,k} = 2^{-3k/4} \int e^{i\langle x,\xi\rangle} \hat{\psi}_{\omega,k}(\xi) \hat{f}(\xi)\, d\xi$$

$$\hat{\psi}_{\omega,k}(\xi)\, \hat{f}(\xi) = 2^{-3k/4} \sum_{x \in \Xi_{\omega,k}} c_{x,\omega,k}\, e^{-i\langle x,\xi\rangle}$$

Reconstruction is periodic, so localize:

$$\hat{\psi}_{\omega,k}(\xi)^2\, \hat{f}(\xi) = 2^{-3k/4} \sum_{x \in \Xi_{\omega,k}} c_{x,\omega,k}\, e^{-i\langle x,\xi\rangle}\, \hat{\psi}_{\omega,k}(\xi)$$

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Sum over ω then k to recover f

$$\hat{f}(\xi) = \sum_k \sum_\omega \sum_{x \in \Xi_{\omega,k}} c_{x,\omega,k}\, 2^{-3k/4} e^{-i\langle x,\xi\rangle} \hat{\psi}_{\omega,k}(\xi)$$

$$c_{x,\omega,k} = \int 2^{-3k/4} e^{i\langle x,\xi\rangle} \hat{\psi}_{\omega,k}(\xi) \hat{f}(\xi)\, d\xi$$

Take inverse Fourier transform:

$$f(y) = \sum_k \sum_\omega \sum_{x \in \Xi_{\omega,k}} c_{x,\omega,k}\, 2^{-3k/4} \psi_{\omega,k}(y - x)$$

$$c_{x,\omega,k} = \int \overline{2^{-3k/4} \psi_{\omega,k}(y - x)}\, f(y)\, dy$$

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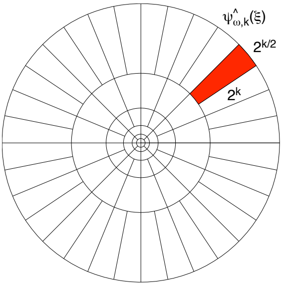
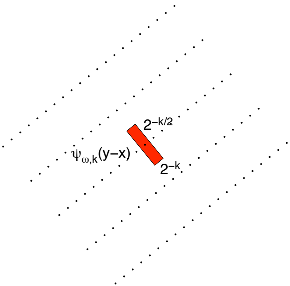
Curvelets: $\varphi_\gamma(y) = 2^{-3k/4}\psi_{\omega,k}(y-x)\,,\quad \gamma=(x,\omega,k)$

Frequency support:



Spatial support:



Plancherel identity

$$\sum_{x\in\Xi_{\omega,k}} |c_{x,\omega,k}|^2 = \int \hat{\psi}_{\omega,k}(\xi)^2 |\hat{f}(\xi)|^2 \, d\xi$$

Sum over ω and k:

$$\sum_{\gamma} |c_\gamma|^2 = \int |\hat{f}(\xi)|^2 \, d\xi = \int |f(y)|^2 \, dy$$

Frame, not basis: $\psi_{\omega,k}(y-x)$ not orthogonal, but almost.

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Motivation for SDD: homogeneous phase functions

Consider $\Phi(\xi)$ smooth for $\xi \neq 0$, homogeneous degree 1:

$$\Phi(s\xi) = s\Phi(\xi)\,, \quad s > 0$$

Example: $\Phi(\xi) = |\xi|$

Claim: *on supp*$(\hat{\psi}_{\omega,k})$, $\quad \Phi(\xi) = \nabla\Phi(\omega)\cdot\xi + r(\xi)\,,$

where

$$|r(\xi)| \lesssim 1$$

$$|\nabla_\omega^m \nabla_{\omega^\perp}^n r(\xi)| \lesssim 2^{-km-\frac{k}{2}n}$$

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Solution to half-wave Cauchy problem:

$$\left(\partial_t + i\sqrt{-\Delta_y}\,\right)u(t,y) = 0\,, \quad u(0,y) = \varphi_\gamma(y)$$

$$\begin{aligned} u(t,y) &= \int e^{i\langle y,\xi\rangle - it|\xi|}\,\hat{\varphi}_\gamma(\xi)\,d\xi \\ &= \int e^{i\langle y - t\omega,\xi\rangle}\,\left[e^{-itr(\xi)}\,\hat{\varphi}_\gamma(\xi)\right]\,d\xi \\ &\approx \varphi_\gamma(y - t\omega) \end{aligned}$$

Initial data: $u(0,y) = \sum_\gamma c_\gamma \varphi_\gamma(y)$, then

$$u(t,y) \approx \sum_\gamma \varphi_\gamma(y - t\omega)$$

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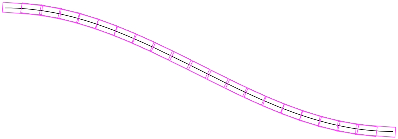
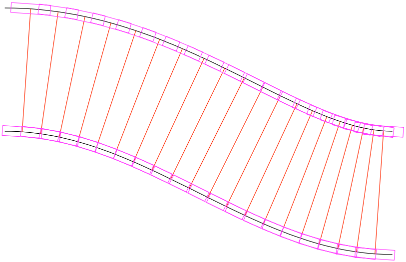
Evolution of waves

A wavefront consisting of a few curvelets:



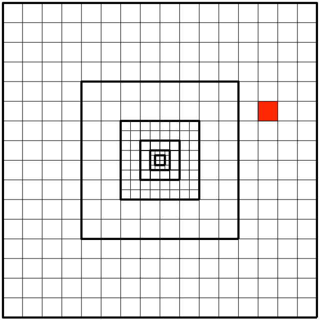
Evolution of waves

First approximation to wave flow:



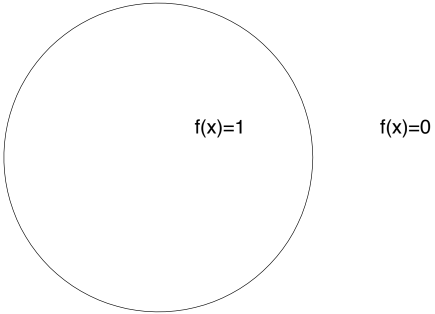
Córdoba-Fefferman Decomposition: $2^{k/2} \times 2^{k/2}$ cubes

Alternative phase-space decomposition being explored to compute wave propagators: (Demanét-Ying)



C. Fefferman [1973]

Decompose support function of unit disc:



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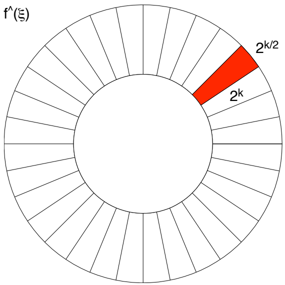
Frequency sectors:



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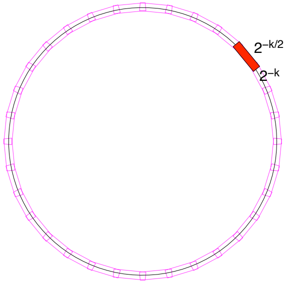
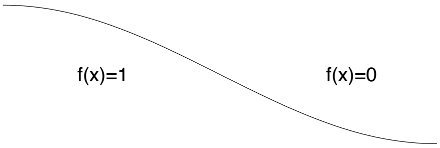


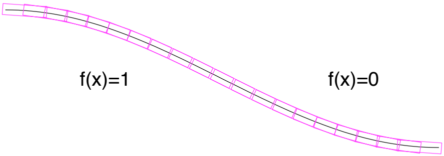
Candés-Donoho (2003)

Image with sharp jump along smooth curve:



Candés-Donoho (2003)

$2^{k/2}$ dominant terms in curvelet expansion at frequency 2^k:



Candés-Donoho (2003)

Approximation rate is optimal:

- Choose n largest coefficients c_γ in $f = \sum_\gamma c_\gamma \varphi_\gamma$

$$\|f - f_n\|_{L^2}^2 \lesssim n^{-2} \log(n)^3$$

- No frame can do better for jumps along C^2 curves.
- Wavelet expansion: $\|f - f_n\|_{L^2}^2 \lesssim n^{-1}$

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Denoising Images with Curvelets