Strichartz Estimates for the Wave Equation on Compact Manifolds with Boundary

Hart F. Smith

Department of Mathematics
University of Washington, Seattle

Carolina Meeting on Harmonic Analysis and PDE

Wave equation on Riemannian manifold (M, g)

Cauchy problem:

$$\partial_t^2 u(t,x) - \Delta_{\mathrm{g}} u(t,x) = 0$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Strichartz estimates:

$$\|u\|_{L^p_t L^q_x((-T,T)\times M)} \lesssim \|\langle D\rangle^\gamma f\|_{L^2(M)} + \|\langle D\rangle^{\gamma-1} g\|_{L^2(M)}$$

For manifolds without boundary:

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \qquad\qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}$$

Wave equation on Riemannian manifold (M, g)

Cauchy problem:

$$\partial_t^2 u(t,x) - \Delta_{\mathrm{g}} u(t,x) = 0$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Strichartz estimates:

$$\|u\|_{L_t^p L_x^q((-T,T)\times M)} \lesssim \|\langle D\rangle^\gamma f\|_{L^2(M)} + \|\langle D\rangle^{\gamma-1} g\|_{L^2(M)}$$

For manifolds without boundary:

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \qquad \frac{2}{p} + \frac{n-1}{q} \le \frac{n-1}{2}$$

Wave equation on Riemannian manifold (M, g)

Cauchy problem:

$$\partial_t^2 u(t,x) - \Delta_{\mathrm{g}} u(t,x) = 0$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Strichartz estimates:

$$\|u\|_{L^p_t L^q_x((-T,T)\times M)} \lesssim \|\langle D\rangle^{\gamma} f\|_{L^2(M)} + \|\langle D\rangle^{\gamma-1} g\|_{L^2(M)}$$

For manifolds without boundary:

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \qquad\qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}$$

Blair-S.-Sogge, 2008

(M, g)= compact manifold with boundary, Dirichlet or Neumann conditions at ∂M, then the Strichartz estimates hold if

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \quad \text{for} \quad \begin{cases} \frac{3}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}, & n \leq 4 \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, & n \geq 4 \end{cases}$$

- Estimates are in subcritical range: small angle gain of $\theta^{\sigma(p,q)}$ cancels $\theta^{-1/p}$ loss in summing over slabs of time length $\Delta t = \theta$.
- For $n \geq 3$ the range of p, q certainly not optimal, but includes important estimates.

Blair-S.-Sogge, 2008

(M,g)= compact manifold with boundary, Dirichlet or Neumann conditions at ∂M, then the Strichartz estimates hold if

$$\frac{1}{p}+\frac{n}{q}=\frac{n}{2}-\gamma \quad \text{for} \quad \begin{cases} \frac{3}{p}+\frac{n-1}{q}\le\frac{n-1}{2}, & n\le 4 \\ \frac{1}{p}+\frac{1}{q}\le\frac{1}{2}, & n\ge 4 \end{cases}$$

- Estimates are in subcritical range: small angle gain of $\theta^{\sigma(p,q)}$ cancels $\theta^{-1/p}$ loss in summing over slabs of time length $\Delta t=\theta$.
- For $n\ge 3$ the range of p,q certainly not optimal, but includes important estimates.

Blair-S.-Sogge, 2008

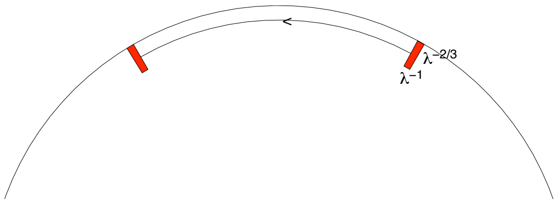
(M, g)= compact manifold with boundary, Dirichlet or Neumann conditions at ∂M, then the Strichartz estimates hold if

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \quad \text{for} \quad \begin{cases} \frac{3}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}, & n \leq 4 \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, & n \geq 4 \end{cases}$$

- Estimates are in subcritical range: small angle gain of $\theta^{\sigma(p,q)}$ cancels $\theta^{-1/p}$ loss in summing over slabs of time length $\Delta t = \theta$.
- For $n \geq 3$ the range of p, q certainly not optimal, but includes important estimates.

Saturating non-example

Gliding wave packet dimensions $\lambda^{-1} \times \lambda^{-2/3}$:



Superimpose gliding eigenfunctions

$$f(r,\theta) = \sum_{n\in[\lambda,2\lambda]} e^{in\theta} J_n(c_{0,n}r)$$

- $J_n(c_{0,n}r) \approx \phi(\lambda^{2/3}(1-r))\,, \qquad n\approx\lambda$
- Associated frequency is $c_{0,n} = n + c_0 n^{1/3} + \dots$

$$\exp(it\sqrt{-\Delta})f \approx \phi(\lambda^{2/3}(1-r)) \sum_{n\in[\lambda,2\lambda]} e^{in(\theta-t)} e^{-ic_0 t\, n^{1/3}}$$

Superimpose gliding eigenfunctions

$$f(r,\theta) = \sum_{n\in[\lambda,2\lambda]} e^{in\theta} J_n(c_{0,n} r)$$

- $J_n(c_{0,n} r) \approx \phi\big(\lambda^{2/3}(1-r)\big)\,, \qquad n \approx \lambda$
- Associated frequency is $c_{0,n} = n + c_0 n^{1/3} + \dots$

$$\exp(it\sqrt{-\Delta})f \approx \phi\big(\lambda^{2/3}(1-r)\big) \sum_{n\in[\lambda,2\lambda]} e^{in(\theta-t)} e^{-ic_0 t\, n^{1/3}}$$

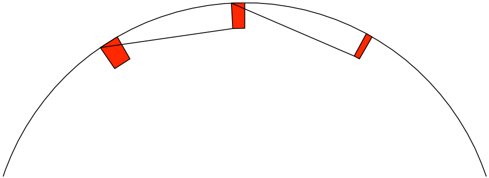
Superimpose gliding eigenfunctions

$$f(r,\theta) = \sum_{n\in[\lambda,2\lambda]} e^{in\theta} J_n(c_{0,n} r)$$

- $J_n(c_{0,n} r) \approx \phi\big(\lambda^{2/3}(1-r)\big)\,, \qquad n \approx \lambda$
- Associated frequency is $c_{0,n} = n + c_0 n^{1/3} + \dots$

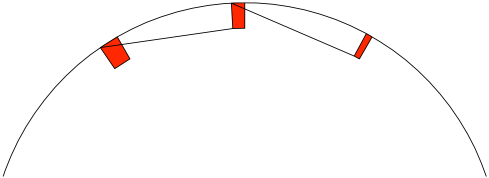
$$\exp(it\sqrt{-\Delta})f \approx \phi\big(\lambda^{2/3}(1-r)\big) \sum_{n\in[\lambda,2\lambda]} e^{in(\theta-t)} e^{-ic_0 t\, n^{1/3}}$$

Dispersion in θ



- Restricted to span of gliding modes $\left\{e^{in\theta}J_n(c_{0,n}r)\right\}_{n\in\mathbb{Z}}$ $\exp(it\sqrt{-\Delta})$ satisfies dispersive/Strichartz estimates.

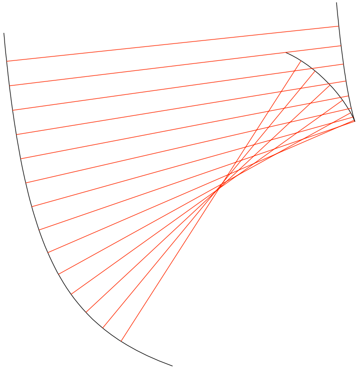
Dispersion in θ



- Restricted to span of gliding modes $\left\{e^{in\theta}J_n(c_{0,n}r)\right\}_{n\in\mathbb{Z}}$ $\exp(it\sqrt{-\Delta})$ satisfies dispersive/Strichartz estimates.

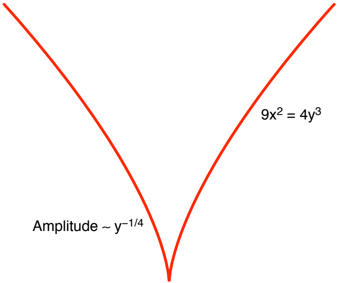
Cusps in wavefronts

Generic wavefronts eventually develop cusps:



Front on model caustic: $x = \pm\frac{2}{3}y^{3/2}$

$$f(x,y) = \int e^{ix\xi + iy\eta - i\frac{1}{3}\eta^3/\xi^2}\,d\eta\,d\xi$$

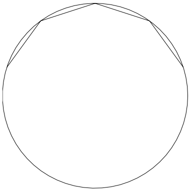


Cusp behavior: Simple vanishing of $d^2\Phi$ at $\eta = 0$

Model for convex boundaries

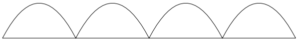
Disc: $r \leq 1$

$$\Delta = \partial_r^2 + \frac{1}{r^2}\partial_\theta^2$$



Model domain: $y \geq 0$

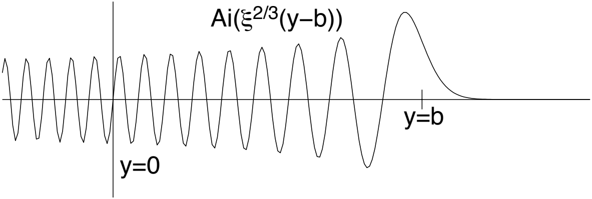
$$\Delta_g = \partial_y^2 + (1+y)\partial_x^2$$



Same to first order under $x = \theta \qquad y = 1 - r$

Eigenfunctions for $\partial_y^2 + (1+y)\partial_x^2$

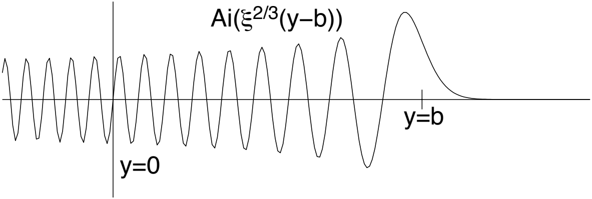
$e^{ix\xi} Ai(\xi^{2/3}(y-b))$ Frequency $= \xi\sqrt{1+b}$



- Boundary condition: $-\xi^{2/3} b =$ zero of Airy function
- Fixed zero of Ai forces $b = c_0\,\xi^{-2/3} \Rightarrow$ dispersion.

Eigenfunctions for $\partial_y^2 + (1+y)\partial_x^2$

$e^{ix\xi} Ai\big(\xi^{2/3}(y-b)\big)$ Frequency $= \xi\sqrt{1+b}$

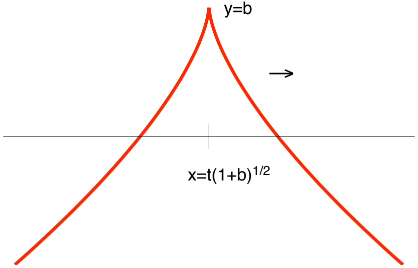


- Boundary condition: $-\xi^{2/3}b =$ zero of Airy function
- Fixed zero of Ai forces $b = c_0\,\xi^{-2/3} \;\Rightarrow\;$ dispersion.

Fix b, ignore boundary condition:

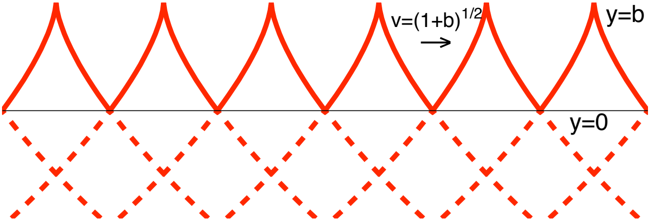
$$\int e^{i\xi(x-t\sqrt{1+b}\,)} Ai\big(\xi^{2/3}(y-b)\big)\,\xi^{-2/3}\,d\xi$$

$$= \int e^{i\xi(x-t\sqrt{1+b}\,)+i\eta(b-y)-i\frac{1}{3}\eta^3/\xi^2}\,d\xi\,d\eta$$

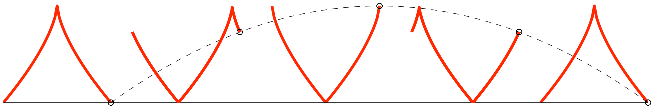


Boundary conditions: $\xi^{2/3}b = -\omega_k \;:\; \xi \approx k\pi \cdot \frac{3}{2}b^{-3/2}$

Poisson summation: periodize by $\frac{4}{3}b^{3/2}$

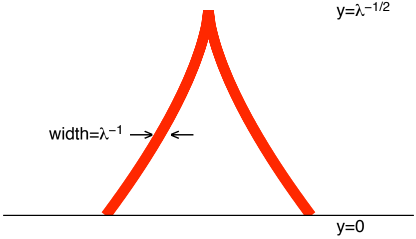


Ray optics tracking of cusp wavefront in domain:



Ivanovici's example:

Single cusp localized to frequency $\xi \in [\lambda, 2\lambda]$ *with* $b = \lambda^{-\frac{1}{2}+\epsilon}$



$$|f_\lambda| \approx \langle \lambda^{2/3}(b-y)\rangle^{-1/4}\,\phi\big(\lambda(x \pm \tfrac{2}{3}(b-y)^{3/2})\big)$$

Ivanovici's example: explicit construction

$$u_\lambda(t,x,y) = \int e^{i\xi(x-t\sqrt{1+b}\,)+i\eta(b-y)-i\frac{1}{3}\eta^3/\xi^2} g_\lambda(t,\xi,\eta)\,d\xi\,d\eta$$

- $g_\lambda = g_{\lambda,1} + g_{\lambda,2} + \cdots$
- $\mathrm{tr}(u_{\lambda,j+1})$ cancels $\mathrm{tr}(u_{\lambda,j})$

Preserving ray-optics picture requires $b \gg \lambda^{-1/2}$.

$$\frac{\|u_\lambda(t,\cdot)\|_q}{\|u_\lambda(0,\cdot)\|_2} \approx \begin{cases} \lambda^{\frac{3}{4}(\frac{1}{2}-\frac{q}{4})} & q < 4 \\ \lambda^{\frac{5}{3}(\frac{1}{2}-\frac{1}{q})-\frac{1}{24}} & q > 4 \end{cases}$$

Ivanovici [2008] ($n = 2$)

Strichartz estimates fail for $\frac{3}{p} + \frac{1}{q} > \frac{15}{24}$

Blair-S-Sogge [2008] ($n = 2$)

Strichartz estimates hold for $\frac{3}{p} + \frac{1}{q} \leq \frac{1}{2}$

May hold with loss of derivatives...

$$\frac{\|u_\lambda(t,\cdot)\|_q}{\|u_\lambda(0,\cdot)\|_2} \approx \begin{cases} \lambda^{\frac{3}{4}(\frac{1}{2}-\frac{q}{4})} & q<4 \\ \lambda^{\frac{5}{3}(\frac{1}{2}-\frac{1}{q})-\frac{1}{24}} & q>4 \end{cases}$$

Ivanovici [2008] ($n=2$)

Strichartz estimates fail for $\frac{3}{p}+\frac{1}{q}>\frac{15}{24}$

Blair-S-Sogge [2008] ($n=2$)

Strichartz estimates hold for $\frac{3}{p}+\frac{1}{q}\le\frac{1}{2}$

May hold with loss of derivatives...

Energy critical wave equation for $n = 3$

$$\Box u(t,x) = -u^5(t,x)\,, \qquad u|_{\partial\Omega} = 0\,,$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Two key Strichartz estimates: $\Box u = 0$

$$\|u\|_{L^5_t L^{10}_x([0,T]\times\Omega)} + \|u\|_{L^4_t L^{12}_x([0,T]\times\Omega)} \lesssim \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$$

- Blair-S-Sogge [2008]: T finite, $\Omega =$ manifold with boundary
- S-Sogge [2000]: Local $\Rightarrow$ global with $T = \infty$ if Ω exterior domain to nontrapping obstacle.

Energy critical wave equation for $n = 3$

$$\Box u(t,x) = -u^5(t,x)\,, \qquad u|_{\partial\Omega} = 0\,,$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Two key Strichartz estimates: $\Box u = 0$

$$\|u\|_{L^5_t L^{10}_x([0,T]\times\Omega)} + \|u\|_{L^4_t L^{12}_x([0,T]\times\Omega)} \lesssim \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$$

- Blair-S-Sogge [2008]: T finite, $\Omega =$ manifold with boundary
- S-Sogge [2000]: Local $\Rightarrow$ global with $T = \infty$
  if Ω exterior domain to nontrapping obstacle.

Applications

- $L_t^5 L_x^{10}$ contraction argument $\Rightarrow$ small data global existence

$$\|u^5\|_{L_t^1 L_x^2} \leq \|u\|^5_{L_t^5 L_x^{10}}$$

- Global existence for large data requires L^6 decay:

$$\lim_{t \to t_0^-} \int_{|x-x_0| \leq t_0 - t} |u|^6 \, dx = 0$$

Grillakis-Shatah-Struwe [1992]: $\Omega = \mathbb{R}^3$ using $L_t^4 L_x^{12}$

Burq-Lebeau-Planchon [2008]: $\partial\Omega$ compact

Applications

- $L^5_t L^{10}_x$ contraction argument $\Rightarrow$ small data global existence

$$\|u^5\|_{L^1_t L^2_x} \leq \|u\|^5_{L^5_t L^{10}_x}$$

- Global existence for large data requires L^6 decay:

$$\lim_{t\to t_0^-} \int_{|x-x_0|\leq t_0-t} |u|^6\, dx = 0$$

Grillakis-Shatah-Struwe [1992]: $\Omega = \mathbb{R}^3$ using $L^4_t L^{12}_x$

Burq-Lebeau-Planchon [2008]: $\partial\Omega$ compact

Applications

- $L^5_t L^{10}_x$ contraction argument $\Rightarrow$ small data global existence

$$\|u^5\|_{L^1_t L^2_x} \le \|u\|^5_{L^5_t L^{10}_x}$$

- Global existence for large data requires L^6 decay:

$$\lim_{t \to t_0^-} \int_{|x-x_0| \le t_0 - t} |u|^6 \, dx = 0$$

Grillakis-Shatah-Struwe [1992]: $\Omega = \mathbb{R}^3$ using $L^4_t L^{12}_x$

Burq-Lebeau-Planchon [2008]: $\partial\Omega$ compact

Scattering for $\Box u = -u^5$

Bahouri-Gerard-Shatah [1998-99]: $\Omega = \mathbb{R}^3$

There exists free solutions $\Box v^{\pm} = 0$ such that

$$\lim_{t \to \pm\infty} E(u - v_{\pm})(t) = 0$$

Key step: $$\lim_{t \to \pm\infty} \int_{\mathbb{R}^3} |u|^6(t,x)\, dx = 0$$

Blair-S-Sogge [2008]: $\Omega =$ exterior to star-shaped obstacle.

Scattering for $\Box u = -u^5$

Bahouri-Gerard-Shatah [1998-99]: $\Omega = \mathbb{R}^3$

There exists free solutions $\Box v^{\pm} = 0$ such that

$$\lim_{t\to\pm\infty} E(u - v_{\pm})(t) = 0$$

Key step: $$\lim_{t\to\pm\infty} \int_{\mathbb{R}^3} |u|^6(t,x)\,dx = 0$$

Blair-S-Sogge [2008]: $\Omega =$ exterior to star-shaped obstacle.

Energy critical wave equation for $n = 4$

$$\Box u(t,x) = -u^3(t,x)\,, \qquad u|_{\partial\Omega} = 0\,,$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Key Strichartz estimate: $\Box u = 0$

$$\|u\|_{L^3_t L^6_x([0,T]\times\Omega)} \lesssim \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$$

- Contraction argument + energy conservation $\Rightarrow$ small data global existence, large data global existence for sub-critical powers.
- Large data global existence for critical power requires an estimate with $p < 3$, but $p = 3$ is endpoint for [B-S-S].

Energy critical wave equation for $n = 4$

$$\Box u(t,x) = -u^3(t,x)\,, \qquad u|_{\partial\Omega} = 0\,,$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Key Strichartz estimate: $\Box u = 0$

$$\|u\|_{L^3_t L^6_x([0,T]\times\Omega)} \lesssim \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$$

- Contraction argument + energy conservation $\Rightarrow$ small data global existence, large data global existence for sub-critical powers.
- Large data global existence for critical power requires an estimate with $p < 3$, but $p = 3$ is endpoint for [B-S-S].

Energy critical wave equation for $n = 4$

$$\Box u(t,x) = -u^3(t,x)\,, \qquad u|_{\partial\Omega} = 0\,,$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Key Strichartz estimate: $\Box u = 0$

$$\|u\|_{L^3_t L^6_x([0,T]\times\Omega)} \lesssim \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$$

- Contraction argument + energy conservation $\Rightarrow$ small data global existence, large data global existence for sub-critical powers.
- Large data global existence for critical power requires an estimate with $p < 3$, but $p = 3$ is endpoint for [B-S-S].