Lp Bounds for Spectral Clusters on Compact Manifolds with Boundary

Hart F. Smith

Department of Mathematics
University of Washington, Seattle

Carolina Meeting on Harmonic Analysis and PDE
$(M, g) = \text{compact 2-d Riemannian manifold}$

- $\Delta_g = \text{Laplacian (Dirichlet or Neumann if } \partial M \neq \emptyset )$
  
  Eigenbasis: $-\Delta_g \phi_j = -\lambda_j^2 \phi_j$ ( $\lambda_j = \text{frequency}$ )

- Spectral Cluster, frequency $\lambda$:
  
  $$f = \sum_{\lambda_j \in [\lambda, \lambda+1]} c_j \phi_j$$

- Goal: find sharp powers $\delta(p)$ such that
  
  $$\frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \lesssim \lambda^{\delta(p)} \quad (p \geq 2)$$
Eigenfunctions and Spectral Clusters
Squarefunction estimates

\((M, g) = \text{compact 2-d Riemannian manifold}\)

\[ \Delta_g = \text{Laplacian ( Dirichlet or Neumann if } \partial M \neq \emptyset ) \]

Eigenbasis: \(-\Delta_g \phi_j = -\lambda_j^2 \phi_j \quad ( \lambda_j = \text{frequency} )\)

Spectral Cluster, frequency \(\lambda\):

\[ f = \sum_{\lambda_j \in [\lambda, \lambda + 1]} c_j \phi_j \]

Goal: find sharp powers \(\delta(p)\) such that

\[ \frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \lesssim \lambda^{\delta(p)} \quad (p \geq 2) \]
\((M, g) = \text{compact 2-d Riemannian manifold}\)

- \(\Delta_g = \text{Laplacian ( Dirichlet or Neumann if } \partial M \neq \emptyset )\)

  Eigenbasis: 
  
  \[-\Delta_g \phi_j = -\lambda_j^2 \phi_j \quad (\lambda_j = \text{frequency})\]

- Spectral Cluster, frequency \(\lambda\):

  \[f = \sum_{\lambda_j \in [\lambda, \lambda+1]} c_j \phi_j\]

- Goal: find sharp powers \(\delta(p)\) such that

  \[
  \frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \lesssim \lambda^{\delta(p)} \quad (p \geq 2)
  \]
Example 1: $f =$ highest weight spherical harmonic.

\[ |f| \approx (1 + \lambda \frac{1}{2} \sin(\phi))^{-N} \]

\[
\frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^{\frac{1}{2}} \left( \frac{1}{2} - \frac{1}{p} \right)
\]

Lower bound (critical region)

\[ \delta(p) \geq \frac{1}{2} \left( \frac{1}{2} - \frac{1}{p} \right) \]
Example 1: $f =$ highest weight spherical harmonic.

$$|f| \approx (1 + \lambda^2 \sin(\phi))^{-N}$$

$$\frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^\frac{1}{2}(\frac{1}{2} - \frac{1}{p})$$

Lower bound (critical region)

$$\delta(p) \geq \frac{1}{2}(\frac{1}{2} - \frac{1}{p})$$
Example 2: \( f = \) zonal spherical harmonic, rotation invariant.

\[
|f| \approx (1 + \lambda \cos(\phi))^{-1/2}
\]

\[
\frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^{\frac{1}{2} - \frac{2}{p}} = \lambda^{2\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}}
\]

Lower bound (sub-critical region)

\[
\delta(p) \geq 2\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}
\]
Example 2: $f = \text{zonal spherical harmonic, rotation invariant.}$

$$|f| \approx (1 + \lambda \cos(\phi))^{-1/2}$$

$$\frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^{1/2 - 2/p} = \lambda^{2(1/2 - 1/p) - 1/2}$$

Lower bound (sub-critical region)

$$\delta(p) \geq 2(1/2 - 1/p) - 1/2$$
Theorem: Sogge [1988]

For compact $n$-dimensional manifold without boundary

$$\delta(p) = \begin{cases} \frac{n-1}{2} \left( \frac{1}{2} - \frac{1}{p} \right), & 2 \leq p \leq \frac{2(n+1)}{n-1} \\ n \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \end{cases}$$

$n = 2$
Grieser [1992]

Sogge’s spectral cluster estimates fail on $D = \{ |x| \leq 1 \} \subseteq \mathbb{R}^2$

**Example:** $f(x) = e^{in\theta} J_n(c_0r), \quad J_n(c_0) = 0$.

$f(x)$ concentrated in $\text{dist}(x, \partial D) \lesssim n^{-\frac{2}{3}}$

\[
\text{Vol("support"}(f)) \approx 1 \times n^{-\frac{2}{3}} \quad \Rightarrow \quad \frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim n^{\frac{2}{3}(\frac{1}{2} - \frac{1}{p})}
\]

Lower bound on manifolds with boundary

\[
\delta(p) \geq \frac{2}{3} \left( \frac{1}{2} - \frac{1}{p} \right)
\]
Grieser [1992]

Sogge’s spectral cluster estimates fail on \( D = \{ |x| \leq 1 \} \subseteq \mathbb{R}^2 \)

Example: \( f(x) = e^{in\theta} J_n(c_0 r), \quad J_n(c_0) = 0. \)

\( f(x) \) concentrated in \( \text{dist}(x, \partial D) \lesssim n^{-\frac{2}{3}} \)

\[ \text{Vol}(\text{“support”}(f)) \approx 1 \times n^{-\frac{2}{3}} \Rightarrow \frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim n^{\frac{2}{3}(\frac{1}{2} - \frac{1}{p})} \]

Lower bound on manifolds with boundary

\[ \delta(p) \geq \frac{2}{3} \left( \frac{1}{2} - \frac{1}{p} \right) \]
Multiple reflections / Gliding rays!

Nondispersive region: angular spread $\approx \lambda^{-\frac{1}{3}}$
physical spread $\approx \lambda^{-\frac{2}{3}}$

Smith-Sogge [1995]
$\partial M$ strictly concave $\Rightarrow$ spectral cluster estimates hold
Multiple reflections / Gliding rays!

Nondispersive region: \(\text{angular spread} \approx \lambda^{-\frac{1}{3}}\)

\(\text{physical spread} \approx \lambda^{-\frac{2}{3}}\)

Smith-Sogge [1995]

\(\partial M\) strictly concave \(\Rightarrow\) spectral cluster estimates hold
Smith-Sogge [2007]: $M=2d$ manifold with boundary

Spectral cluster estimates hold with

$$\delta(p) = \begin{cases} \frac{2}{3} \left( \frac{1}{2} - \frac{1}{p} \right), & 6 \leq p \leq 8 \\ 2 \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{2}, & 8 \leq p \leq \infty \end{cases}$$

\[0\quad \frac{1}{4} \quad 1/2\]

\[0\quad 1/8 \quad 1/2\]
Key step in proof: Eliminate the boundary

Geodesic normal coordinates along $\partial M : M = \{ x_2 \geq 0 \}$

$$g = d_{x_2}^2 + a_{11}(x_1, x_2) d_{x_1}^2, \quad \text{smooth on } x_2 \geq 0$$

Extend $g$ across $\partial M$ to be even in $x_2$

$$\tilde{g} = d_{x_2}^2 + a_{11}(x_1, |x_2|) d_{x_1}^2$$

- Odd extension of Dirichlet eigenfunction: $\frac{x_2}{|x_2|} \phi_j(x_1, |x_2|)$ is eigenfunction for $\Delta_{\tilde{g}}$

- Even extension of Neumann eigenfunction: $\phi_j(x_1, |x_2|)$ is eigenfunction for $\Delta_{\tilde{g}}$
Key step in proof: Eliminate the boundary

Geodesic normal coordinates along $\partial M : M = \{ x_2 \geq 0 \}$

$$g = d_{x_2}^2 + a_{11}(x_1, x_2) d_{x_1}^2,$$ smooth on $x_2 \geq 0$

Extend $g$ across $\partial M$ to be even in $x_2$

$$\tilde{g} = d_{x_2}^2 + a_{11}(x_1, |x_2|) d_{x_1}^2$$

- Odd extension of Dirichlet eigenfunction: $\frac{x_2}{|x_2|} \phi_j(x_1, |x_2|)$ is eigenfunction for $\Delta_{\tilde{g}}$

- Even extension of Neumann eigenfunction: $\phi_j(x_1, |x_2|)$ is eigenfunction for $\Delta_{\tilde{g}}$
No more boundary / reflected geodesics:

Disc: $r \leq 1$

$$g = d_r^2 + \frac{1}{r^2} d_\theta^2$$

Normal coordinates: $x_2 = 1 - r$

$$\tilde{g} = d_{x_2}^2 + \frac{1}{(1-|x_2|)^2} d_{x_1}^2$$

But metric is Lipschitz.
Disc: $r \leq 1$

$$g = d_r^2 + \frac{1}{r^2} d_\theta^2$$

Normal coordinates: $x_2 = 1 - r$

$$\tilde{g} = d_{x_2}^2 + \frac{1}{(1-|x_2|)^2} d_{x_1}^2$$

But metric is Lipschitz.
Metric \( \tilde{g} \) is of special Lipschitz type:

\[ d_{x}^{2} \tilde{g} \approx \delta(x_{2}) \]

is integrable along non-tangential geodesics.

\[
\frac{dx_{2}}{dt} \approx \theta \quad \text{on} \quad \gamma
\]

\[
\int d_{x}^{2} \tilde{g}(\gamma(t)) \, dt \approx \theta^{-1}
\]
Consider time dependent metric $g(t, x)$ on $M$

$$\partial_t^2 u(t, x) - \Delta_g u(t, x) = 0$$

$$u(0, x) = f(x), \quad \partial_t u(0, x) = g(x)$$

**Tataru [2002] : Strichartz estimates**

If $\|\nabla_{t,x}^2 g\|_{L^1_tL^\infty_x} \leq 1$, then

$$\|u\|_{L^p_tL^q_x([-1,1] \times M)} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

for same $p, q, s$ as smooth manifolds, Euclidean space.
In our case $\int d^2g \approx \theta^{-1}$

Rescaled metric $g(\theta t, \theta x) \in L^1_t L^\infty_x$ norm 1:

Tataru [2002] : Strichartz estimates

If $\|\nabla^2_{t,x} g\|_{L^1_t L^\infty_x} \leq \theta^{-1}$, then

$$\|u\|_{L^p_t L^q_x([-\theta, \theta] \times M)} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

for same $p, q, s$ as smooth manifolds, Euclidean space.
\[ \| \cos(t \sqrt{-\Delta_g}) f(x) \|_{L^p_x L^2_t(M \times [-1,1])} \lesssim \| \langle D \rangle^{\delta(p)} f \|_{L^2(M)}, \quad p \geq 6 \]

**Squarefunction estimates \Rightarrow spectral cluster bounds:**

For spectral cluster \( f : \cos(t \sqrt{-\Delta_g}) f(x) \approx \cos(t \lambda) f(x) \)

\[ \| f \|_{L^p(M)} \lesssim \| \cos(t \sqrt{-\Delta_g}) f(x) \|_{L^p_x L^2_t(M \times [-1,1])} \lesssim \lambda^{\delta(p)} \| f \|_{L^2(M)} \]

[Mockenhaupt-Seeger-Sogge (1993)] [S., \( d^2 g \in L^1_t L^\infty_x \) (2006)]
Squarefunction estimates

\[ \| \cos(t \sqrt{-\Delta_g}) f(x) \|_{L^p_x L^2_t(M \times [-1,1])} \lesssim \| \langle D \rangle^{\delta(p)} f \|_{L^2(M)}, \quad p \geq 6 \]

Squarefunction estimates ⇒ spectral cluster bounds:

For spectral cluster \( f \):

\[ \cos(t \sqrt{-\Delta_g}) f(x) \approx \cos(t \lambda) f(x) \]

\[ \| f \|_{L^p(M)} \lesssim \| \cos(t \sqrt{-\Delta_g}) f(x) \|_{L^p_x L^2_t(M \times [-1,1])} \lesssim \lambda^{\delta(p)} \| f \|_{L^2(M)} \]

[Mockenhaupt-Seeger-Sogge (1993)] [S., \( d^2 g \in L^1_x L^\infty_t (2006)\)]
Squarefunction estimates

\[ \| \cos(t \sqrt{-\Delta_g}) f(x) \|_{L^p_x L^2_t(M \times [-1,1])} \lesssim \| \langle D \rangle^\delta \|_{L^2(M)}, \quad p \geq 6 \]

Squarefunction estimates ⇒ spectral cluster bounds:

For spectral cluster \( f \) : 
\[ \cos(t \sqrt{-\Delta_g}) f(x) \approx \cos(t \lambda) f(x) \]

\[ \| f \|_{L^p(M)} \lesssim \| \cos(t \sqrt{-\Delta_g}) f(x) \|_{L^p_x L^2_t(M \times [-1,1])} \lesssim \lambda^\delta \| f \|_{L^2(M)} \]

[Mockenhaupt-Seeger-Sogge (1993)] [S., \( d^2 g \in L^1_t L^{\infty}_x \) (2006)]
Phase-space localized spectral clusters:

If $\hat{f}(\xi_1, \xi_2)$ is localized to $\xi_2/\xi_1 \in [\theta, 2\theta]$, then we can prove "good" bounds on $\|f\|_{L^p}$ over slabs $S$ of size $\theta$ in $x_1$ direction.

Problem: add up over $\theta^{-1}$ slabs $\Rightarrow$ lose $\theta^{-1/p}$ for $\|f\|_{L^p(M)}$.  

Hart F. Smith  
Lp Bounds for Spectral Clusters
Phase-space localized spectral clusters:

If $\hat{f}(\xi_1, \xi_2)$ is localized to $\xi_2/\xi_1 \in [\theta, 2\theta]$, then we can prove "good" bounds on $\|f\|_{L^p}$ over slabs $S$ of size $\theta$ in $x_1$ direction.

Problem: add up over $\theta^{-1}$ slabs $\Rightarrow$ lose $\theta^{-1/p}$ for $\|f\|_{L^p(M)}$. 
For subcritical $p > 6$, gain from small angle localization

- If $\hat{f}_\theta$ is localized to a cone of angle $\theta$, then

$$
\| f_\theta \|_{L^p(S)} \lesssim \theta^{\frac{1}{2} - \frac{3}{p}} \lambda^{\delta(p)} \| f_\theta \|_{L^2(M)}
$$

- Combined gain · loss for $f_\theta$

$$
\| f_\theta \|_{L^p(M)} \lesssim \theta^{\frac{1}{2} - \frac{4}{p}} \lambda^{\delta(p)} \| f_\theta \|_{L^2(M)}
$$

- Sum over dyadic decomp in $\theta \leq 1$ yields

$$
\| f \|_{L^p(M)} \lesssim \lambda^{\delta(p)} \| f \|_{L^2(M)}, \quad p \geq 8
$$
For subcritical $p > 6$, gain from small angle localization

- If $\hat{f}_\theta$ is localized to a cone of angle $\theta$, then
  \[
  \|f_\theta\|_{L^p(S)} \lesssim \theta^{\frac{1}{2} - \frac{3}{p}} \lambda^{\delta(p)} \|f_\theta\|_{L^2(M)}
  \]

- Combined gain · loss for $f_\theta$
  \[
  \|f_\theta\|_{L^p(M)} \lesssim \theta^{\frac{1}{2} - \frac{4}{p}} \lambda^{\delta(p)} \|f_\theta\|_{L^2(M)}
  \]

- Sum over dyadic decomp in $\theta \leq 1$ yields
  \[
  \|f\|_{L^p(M)} \lesssim \lambda^{\delta(p)} \|f\|_{L^2(M)}, \quad p \geq 8
  \]
For subcritical $p > 6$, gain from small angle localization

- If $\hat{f}_\theta$ is localized to a cone of angle $\theta$, then
  \[
  \| f_\theta \|_{L^p(S)} \lesssim \theta^{1 - \frac{3}{p}} \lambda^\delta(p) \| f_\theta \|_{L^2(M)}
  \]

- Combined gain · loss for $f_\theta$
  \[
  \| f_\theta \|_{L^p(M)} \lesssim \theta^{1 - \frac{4}{p}} \lambda^\delta(p) \| f_\theta \|_{L^2(M)}
  \]

- Sum over dyadic decomp in $\theta \leq 1$ yields
  \[
  \| f \|_{L^p(M)} \lesssim \lambda^\delta(p) \| f \|_{L^2(M)} , \quad p \geq 8
  \]
Gliding modes: $\theta = \lambda^{-1/3}$

- On slab $S$ size $\lambda^{-1/3}$ in $x_1$:

$$\|f\|_{L^6(S)} \leq \lambda^{1/6} \|f\|_{L^2(M)}$$

Sum over slabs:

$$\|f\|_{L^6(M)} \leq \lambda^{1/6+1/18} \|f\|_{L^2(M)}$$
Gliding modes: $\theta = \lambda^{-1/3}$

- On slab $S$ size $\lambda^{-1/3}$ in $x_1$:

$$\|f\|_{L^8(S)} \leq \lambda^{1/4} \lambda^{-1/24} \|f\|_{L^2(M)}$$

Sum over slabs:

$$\|f\|_{L^8(M)} \leq \lambda^{1/4} \|f\|_{L^2(M)}$$
Higher dimensional results: $n \geq 3$

Smith-Sogge [2007]: $M = n$ dimensional manifold with boundary

No-loss square function / spectral cluster estimates hold with

$$\delta(p) = n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}$$

$$\begin{cases}
  5 \leq p \leq \infty, & n = 3 \\
  4 \leq p \leq \infty, & n \geq 4
\end{cases}$$

Result non-optimal: ignores dispersion tangent to $\partial M$. 