On the Evolution of Curvelets by the Wave Equation

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1st PRIMA Congress

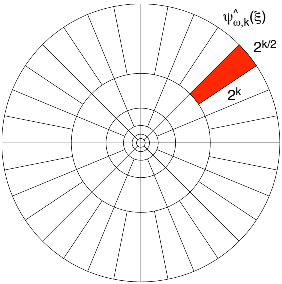
Curvelets

- A curvelet frame $\{\varphi_\gamma\}$ is a wave packet frame on $L^2(\mathbb{R}^2)$ based on second dyadic decomposition.

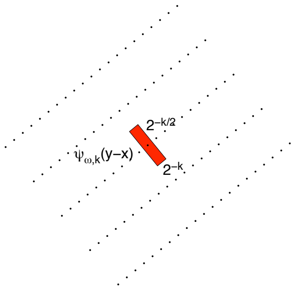
$$
\begin{aligned}
f(x) &= \sum_\gamma c_\gamma \, \varphi_\gamma(x) \\
c_\gamma &= \int f(x) \, \overline{\varphi_\gamma(x)} \, dx
\end{aligned}
$$

Curvelets: $\varphi_\gamma(y) = 2^{-(n+1)k/4}\psi_{\omega,k}(y-x)$

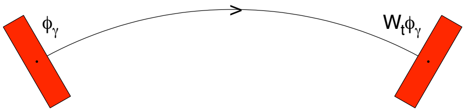
Frequency support:



Spatial support:



Curvelets are “coherent” wave-packets



Linearization of phase functions

$$W_t\varphi_\gamma(x) = \int a(t,x,\xi)\, e^{iS(t,x,\xi)}\hat{\varphi}_\gamma(\xi)\, d\xi$$

- On second dyadic sector: $S(t,x,\xi) \approx \Theta_t(x-\overline{x})\cdot\xi$

$$W_t\varphi_\gamma(x) \approx a_t \cdot \varphi_\gamma(\Theta_t(x-\overline{x}))$$

- Second dyadic decomposition of frequency space:

Largest sectors on which phase functions well-approximated by linear phase functions (up to bounded error)

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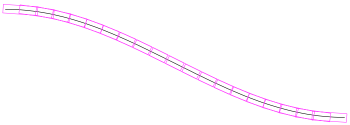
Smith (1998)

Use curvelets to construct wave evolution

A wavefront consisting of a few curvelets:



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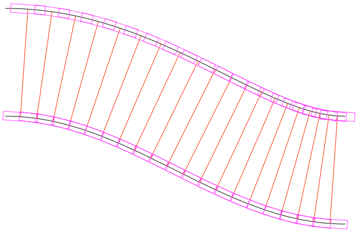
Use curvelets to construct wave evolution

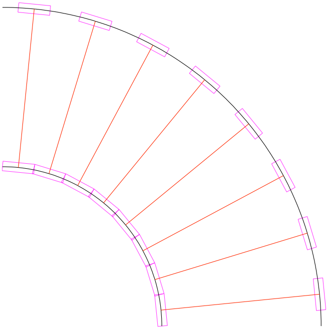
First approximation to wave flow:



Iteration ⇒ exact solution.

Problem: first approximation breaks down for $t \gtrsim 1$

Gaps develop, no curvature.



Curvature, dispersion/spreading are second order terms

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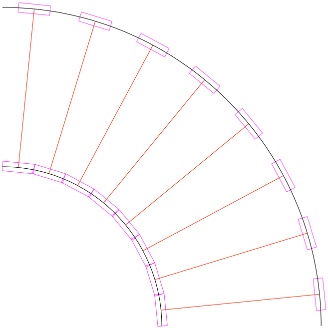

Curvature, dispersion/spreading are second order terms

Quadratic approximation = error of size $2^{-k/2}$

$$W_t\varphi_\gamma(x) = \int a_t(x,\xi)\, e^{iS_t(x,\xi)}\hat{\varphi}_\gamma(\xi)\, d\xi$$

Homogeneous quadratic expansion: $S_t(x,\xi) = \xi_1\, S_t(x,1,\xi'/\xi_1)$

$$S_t(x,\xi) \approx \langle T_t(x-\overline{x}),\xi\rangle + \langle M_t(x-\overline{x}), x-\overline{x}\rangle\xi_1 + \langle Q_t\,\xi',\,\xi'\rangle\xi_1^{-1}$$

$$W_t\varphi_\gamma(x) \approx a_t \int e^{i\langle y_t(x),\xi\rangle - i\xi_1^{-1}\langle Q_t\,\xi',\xi'\rangle}\hat{\varphi}_\gamma(\xi)\, d\xi$$

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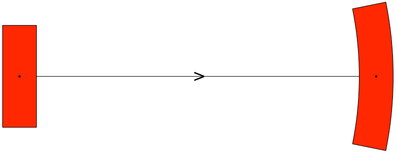
Constant coefficient: $W_t = \exp(it\sqrt{-\Delta}\,)$

$$W_t\varphi_\gamma(x) \approx \int e^{i\langle x - te_1,\xi\rangle - it\xi_1^{-1}|\xi'|^2}\,\hat{\varphi}_\gamma(\xi)\,d\xi$$



Parallel frame along $(\overline{x}, \overline{\xi}) : \partial_t \Theta_t = \Theta_t \cdot p_{\xi x}(t, \overline{x}, \overline{\xi})$

Seek solution mod $2^{-k/2}$: $W_t \varphi_\gamma = \psi_\gamma(t, \Theta_t(x - \overline{x}))$

$$D_t \psi_\gamma = \tfrac{1}{2}\langle A(t)\, x', x'\rangle D_1 \psi_\gamma + \tfrac{1}{2}\langle B(t)\, D', D'\rangle D_1^{-1} \psi_\gamma$$

Admits exact solution for short time:

$$\psi_\gamma(t, x) = b_t \int e^{i\langle T_t\, x, \xi\rangle + i\langle M_t\, x', x'\rangle \xi_1 + i\langle Q_t\, \xi', \xi'\rangle \xi_1^{-1}}\, \hat{\psi}_\gamma(\xi)\, d\xi$$

$$\partial_t T_t + T_t\, B(t)\, M_t = 0$$

$$\partial_t M_t + M_t\, B(t)\, M_t = -A(t)$$

$$\partial_t Q_t + T_t\, B(t)\, T_t' = 0$$

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Hamiltonian flow for $\tau + \frac{1}{2}\langle A(t)\,x', x'\rangle + \frac{1}{2}\langle B(t)\,\xi', \xi'\rangle$

- $$\partial_t\, x' = B(t)\,\xi' \qquad \partial_t\,\xi' = -A(t)\,x'$$

  $$\text{Linear :}\quad \begin{pmatrix} x' \\ \xi' \end{pmatrix} = \begin{pmatrix} W_1 & W_2 \\ W_3 & W_4 \end{pmatrix} \begin{pmatrix} x'_0 \\ \xi'_0 \end{pmatrix}$$

- $$T_t = W_1^{-1}\,, \qquad M_t = W_3\,W_1^{-1}\,, \qquad Q_t = -W_1^{-1}\,W_2$$

$$\tilde{S}(t,x,\xi) \approx \langle T_t\,x,\,\xi\rangle + \langle M_t\,x',\,x'\rangle\xi_1 + \langle Q_t\,\xi',\,\xi'\rangle\xi_1^{-1}$$

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Conjugate point: W_1^{-1} not defined

- Example: $D_t + \frac{1}{2}|x'|^2 D_1 + \frac{1}{2}|D'|^2 D_1^{-1}$

$$\tilde{S}(t,x,\xi) = x_1\,\xi_1 + \sec t\,\langle x',\,\xi'\rangle - \tan t\,|x'|^2\xi_1 - \tan t\,|\xi'|^2\xi_1^{-1}$$

- $t = \frac{\pi}{2}$, Hamiltonian flow : $(x', \xi') \to (\xi'/\xi_1, -x'\xi_1)$

$$W_{\frac{\pi}{2}}\varphi_\gamma(x) \approx \frac{1+i}{\sqrt{2}} \int e^{ix_1\xi_1}\,\hat{\varphi}(\xi_1, -x'\xi_1)\,d\xi_1$$

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Change of variable $\Lambda_{M_0}(x) = \left(x_1 - \langle M_0 x', x' \rangle\,,\; x'\right)$ eliminates conjugate point

$$W_t \varphi_\gamma(x) \approx a_t \cdot \left[\Lambda_{M_t} \circ \exp\left(iQ_t(D', D')/D_1\right) \circ \Lambda_{M_0}\, \varphi_\gamma\right]\left(T_t\, \Theta_t(x - \overline{x})\right)$$

- *All terms can be selected based on linearized Hamiltonian flow about center of φ_γ.*

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