Cubic Scaling for Caustics and Tangential Reflections

Hart F. Smith

Department of Mathematics
University of Washington, Seattle

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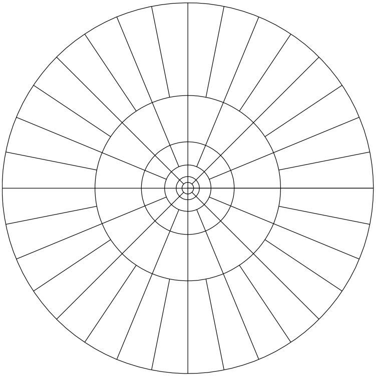
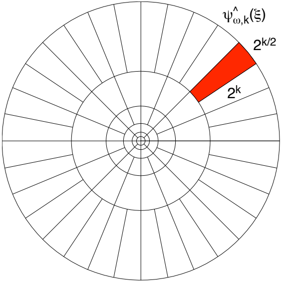
Second Dyadic Decomposition

Frequency sectors: $2^k < |\xi| < 2^{k+1}, \quad \angle(\omega, \xi) \le 2^{-k/2}$

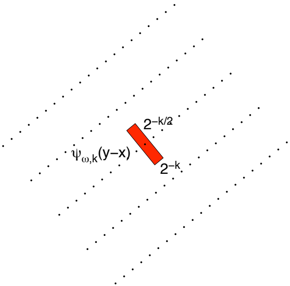


Curvelets: $\varphi_\gamma(y) = 2^{-3k/4}\psi_{\omega,k}(y-x)$

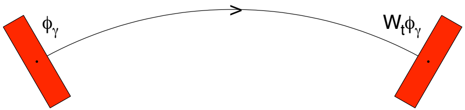
Frequency support:



Spatial support:



Curvelets are “coherent” wave-packets



Linearization of phase functions

$$W_t\varphi_\gamma(x) = \int e^{i\phi(t,x,\xi)}\widehat{\varphi}_\gamma(\xi)\,d\xi$$

- On second dyadic sector: $\phi(t,x,\xi) \approx y_t(x)\cdot\xi$

$$W_t\varphi_\gamma(x) \approx \varphi_\gamma(y_t(x))$$

- Second dyadic decomposition of frequency space:

Largest sectors on which standard phase functions well-approximated by linear phase functions.

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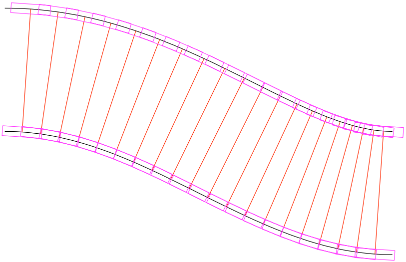
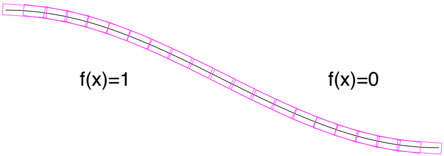
Smith (1998)
Use curvelets to construct wave evolution

First approximation to wave flow:



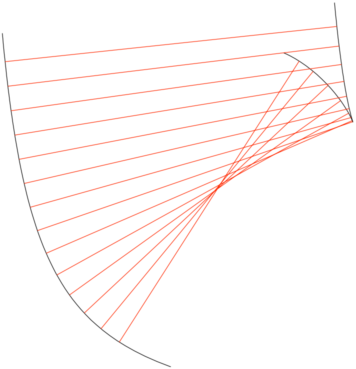
Candés-Donoho (2003)

Optimal approximation to images with jump along C^2 curves



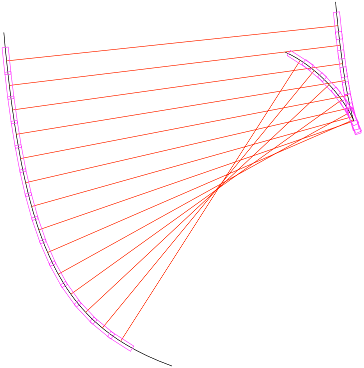
Caustics in wavefronts

Generic wavefronts eventually develop caustics:



Caustics in wavefronts

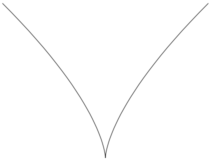
Curvelet flow approximation: high overlap at caustic point



Front on model caustic: $x = -\frac{1}{3}s^3$, $y = \frac{1}{2}s^2$

$$f(x) = \int e^{ix\cdot\xi - i\Phi(\xi)}\, a(\xi)\, d\xi$$

$$\Phi(\xi) = \tfrac{1}{6}\,\xi_1\left(\frac{\xi_2}{\xi_1}\right)^3 \qquad s = \frac{\xi_2}{\xi_1}$$



Caustic behavior: $d^2\Phi = 0$ at $\xi_2 = 0$

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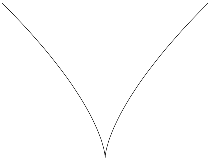


Caustic behavior: $d^2\Phi = 0$ at $\xi_2 = 0$

Phase $\Phi(\xi)$ linearizes on larger sectors:

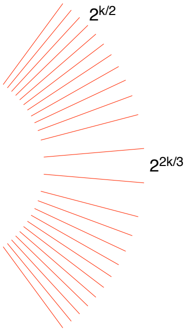
Linearization condition: $d^2\Phi(\xi)(\Delta\xi)^2 \leq 1$

- Near $\xi_2 = 0$, cubic:

$$\Delta\xi = 2^{2k/3}$$

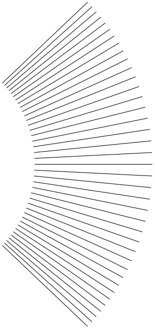
- Near $\xi_2 = \xi_1$, parabolic:

$$\Delta\xi = 2^{k/2}$$

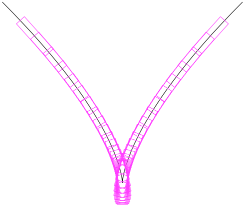


Parabolic Scaling

Frequency sectors:

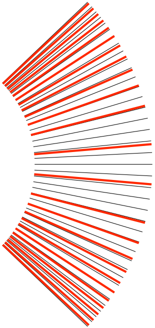


Spatial decomposition:



Interpolated Cubic/Parabolic Scaling

Frequency sectors:



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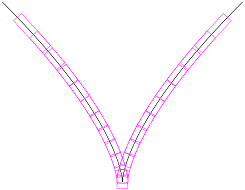
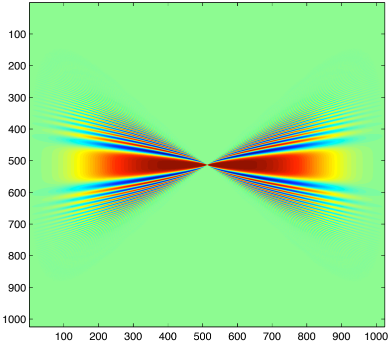
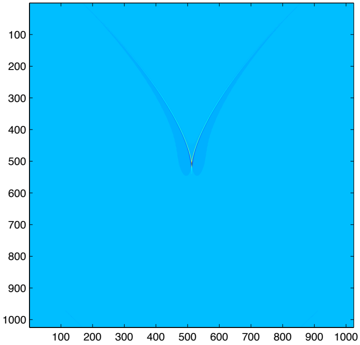


Caustic

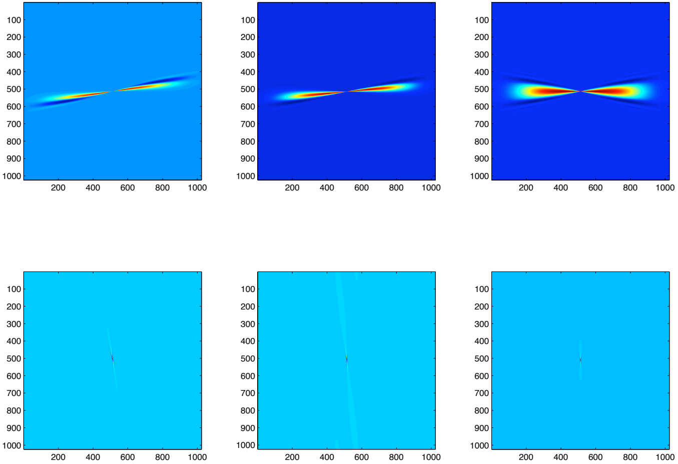
$$\hat{f}(\xi) = exp(i\Phi(\xi))$$



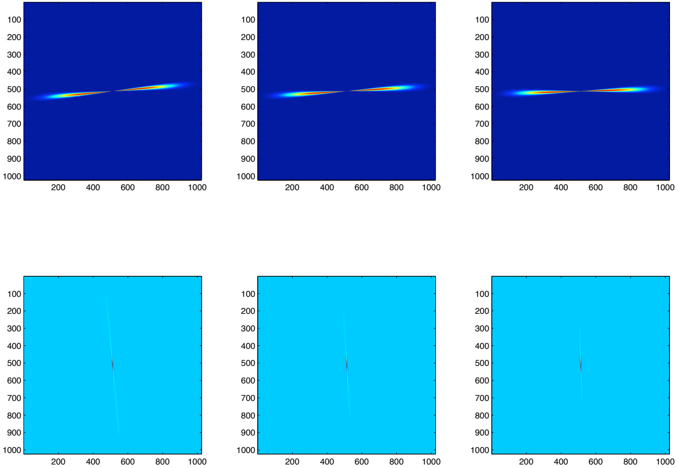
$$f(x)$$



Cubic/Parabolic decomposition of caustic:

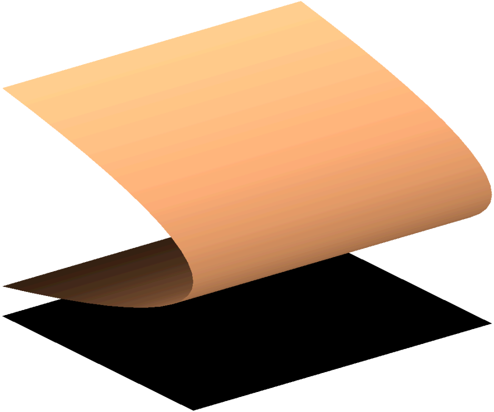


Parabolic decomposition of caustic:



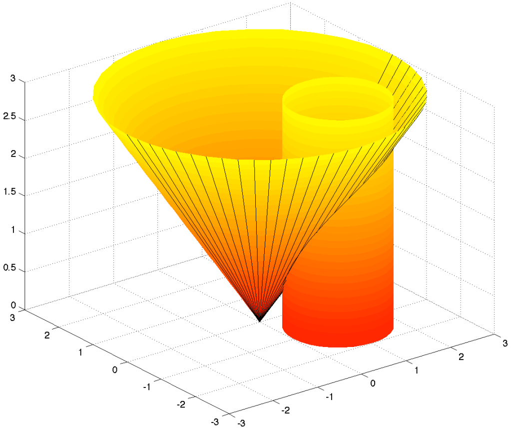
Folding Fourier integral operator

Projection $\Pi_x : \Lambda \to T^*(X)$ has folding singularity:



Example of Folding FIO

Restrict solution to wave equation to convex obstacle:



Model Folding FIO

- Convolution with line measure on cubic:

$$Tf(x_1, x_2) = \int f(x_1 - \tfrac{1}{3}t^3, x_2 - t)\, dt$$

$$\widehat{Tf}(\xi) = \xi_1^{-\frac{1}{3}} Ai(\xi_1^{-\frac{1}{3}}\xi_2)\, \widehat{f}(\xi)$$

- General folding FIO kernel:

$$K(x,y) = \int e^{i\varphi_1(x,\xi)}\, a(x,\xi)\, Ai(\xi_1^{-\frac{1}{3}}\xi_2)\, \overline{b}(y,\xi)\, e^{-i\varphi_2(y,\xi)}\, d\xi$$

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Oscillations $exp(\pm i\frac{2}{3}\xi_1^{-1/2}\xi_2^{3/2})$ of $Ai(\xi_1^{-1/3}\xi_2)$ linearize on smaller sectors:

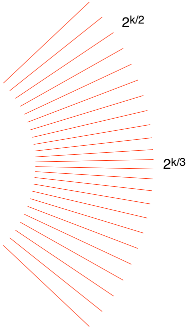
Linearization condition: $d^2\left[\xi_1^{-1/2}\xi_2^{3/2}\right](\Delta\xi)^2 \leq 1$

- Near $\xi_2 = 0$, sub-cubic:

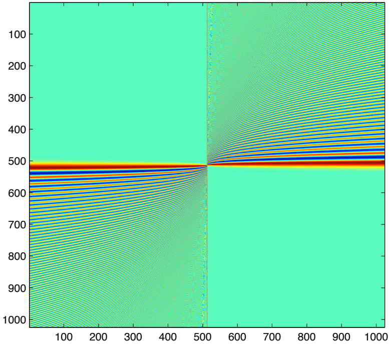
$$\Delta\xi = 2^{k/3}$$

- Near $\xi_2 = \xi_1$, parabolic:

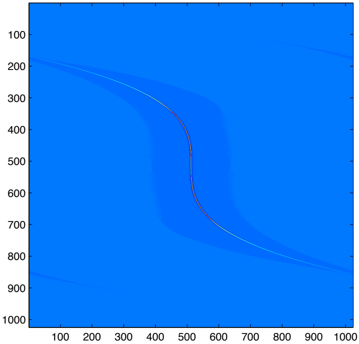
$$\Delta\xi = 2^{k/2}$$



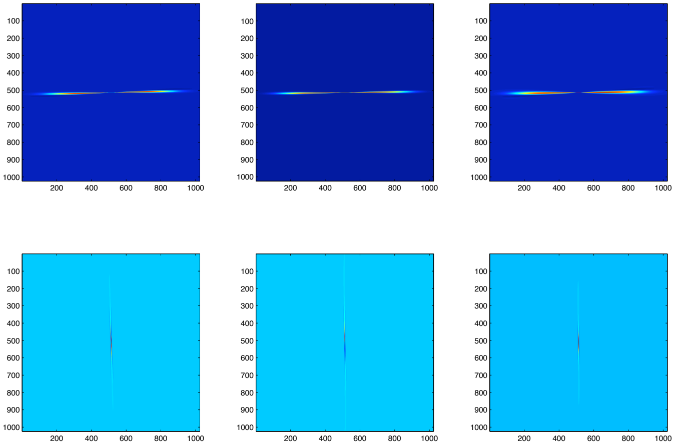
$$Ai(\xi_1^{-1/3}\xi_2)$$



$$x_2 = \frac{1}{3}x_1^3$$



Cubic/Parabolic decomposition of cubic:



Fold $\longleftrightarrow$ Caustic duality: