POINTWISE BOUNDS ON QUASIMODES OF SEMICLASSICAL SCHRÖDINGER OPERATORS IN DIMENSION TWO

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Abstract. We prove optimal pointwise bounds on quasimodes of semiclassical Schrödinger operators with arbitrary smooth real potentials in dimension two. This end-point estimate was left open in the general study of semiclassical $L^p$ bounds conducted by Koch-Tataru-Zworski [2]. However, we show that the results of [2] imply the two dimensional end-point estimate by scaling and localization.

1. Introduction

Let $g_{ij}(x)$ be a positive definite Riemannian metric on $\mathbb{R}^2$ with the corresponding Laplace-Beltrami operator,

$$\Delta_g u := \frac{1}{\sqrt{\bar{g}}} \sum_{i,j} \partial_{x_j} \left( g^{ij} \sqrt{\bar{g}} \partial_{x_i} u \right), \quad (g^{ij}) := (g_{ij})^{-1}, \quad \bar{g} := \det(g_{ij}),$$

and let $V \in C^\infty(\mathbb{R}^2)$ be real valued. We prove the following general bound which was already established (under an additional necessary condition) in higher dimensions in [2], but which was open in dimension two:

**Theorem 1.1.** Suppose that $h \leq 1$, and $u \in H^2_{loc}(\mathbb{R}^2)$. Suppose that $u$ satisfies

$$\| -h^2 \Delta_g u + Vu \|_{L^2} \leq h, \quad \|u\|_{L^2} \leq 1.$$ 

Then for all $K \subseteq \mathbb{R}^2$,

$$\sup_{x \in K} |u(x)| \leq C_K h^{-\frac{1}{2}},$$

where the constant $C_K$ depends only on $g$, $V$, and $K$.

A function $u$ satisfying (1.1) is sometimes called a weak quasimode. It is a local object in the sense that if $u$ is a weak quasimode then $\psi u$, $\psi \in C^\infty_c(\mathbb{R}^2)$ is also one, so the theorem is easily reformulated with $g$, $V$, and $u$ defined on an open subset of $\mathbb{R}^2$. The localization is also valid in phase space: for instance if $\chi \in C^\infty_c(\mathbb{R}^2 \times \mathbb{R}^2)$ then $\chi^w(x, hD)u$ is also a weak

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quasimode – see [1, Chapter 7] or [4, Chapter 4] for the review of the Weyl quantization $\chi \mapsto \chi^w$.

If $\liminf_{|x|\to\infty} V > 0$, then $-h^2\Delta + V$ (defined on $C_c^\infty(\mathbb{R}^2)$) is essentially self-adjoint and the spectrum of $-h^2\Delta + V$ is discrete in a neighbourhood of 0 – see for instance [1, Chapter 4]. In this case weak quasimodes arise as spectral clusters:

\begin{equation}
(1.3) \quad u = \sum_{|E_j| \leq Ch} c_j w_j, \quad (-h^2\Delta + V)w_j = E_j w_j, \quad \langle w_j, w_k \rangle_{L^2} = \delta_{jk}, \quad \sum_j |c_j|^2 \leq 1.
\end{equation}

Then $u$ is a weak quasimode in the sense of (1.1). Since $V(x) \geq c_0 > 0$ for $|x| \geq R$, Agmon estimates (see for instance [1, Chapter 6]) and Sobolev embedding show that $|u(x)| \leq e^{-c_1/h}$, $c_1 > 0$, for $|x| \geq R$. Hence we get global bounds

\[ |u(x)| \leq Ch^{-\frac{1}{2}}, \quad x \in \mathbb{R}^2. \]

It should be stressed however that a weak quasimode is a more general notion than a spectral cluster.

The result also holds when $\mathbb{R}^2$ is replaced by a two dimensional manifold and, as in the example above, gives global bounds on spectral clusters (1.3) when the manifold is compact. If $V < 0$ this is also a by-product of of the Avakumovic-Levitan-Hörmander bound on the spectral function – see [3], and for a simple proof of a semiclassical generalization see [2, §3] or [4, §7.4].

In higher dimensions the theorem requires an additional phase space localization assumption and is a special case of [2, Theorem 6]: Suppose $p(x, \xi)$ is a function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying $\partial_x^\alpha \partial_\xi^\beta p(x, \xi) = \mathcal{O}(|\xi|^m)$ for some $m$. Suppose that $K \Subset \mathbb{R}^n$ and $\chi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and that for $(x, \xi) \in \text{supp} \chi$

\[ p(x, \xi) = 0, \quad d_\xi p(x, \xi) = 0 \implies d_\xi^2 p(x, \xi) \text{ is nondegenerate}. \]

Then for $u(h)$ such that

\begin{equation}
(1.4) \quad \text{supp } u(h) \subset K, \quad u(h) = \chi^w(x, hD)u + \mathcal{O}(h^\infty),
\end{equation}

we have

\begin{equation}
(1.5) \quad \|u(h)\|_\infty \leq C h^{-\frac{n-1}{2}} \left( \|u(h)\|_{L^2} + \frac{1}{h} \|p^w(x, hD)u\|_{L^2} \right), \quad n \geq 3.
\end{equation}

When $n = 2$ the bound holds with $(\log(1/h)/h)^{\frac{1}{2}}$, which is optimal in general if $d_\xi^2 p$ is not positive definite – see [2, §3, §6] and §3 below for examples.

A small bonus for Schrödinger operators in dimension two is the fact that the frequency localization condition in (1.4) required for (1.5) is not necessary – see (2.5) below. And as noted already, in all dimensions the compact support condition on $u$ is easily dropped when working with local estimates on $u$. 
The proof of Theorem 1.1 is reduced to a local result presented in Proposition 2.1. That result follows in turn from a rescaling argument involving several cases, some of which use the following result that forms part of [2, Corollary 1].

**Theorem 1.2.** Suppose that $u = u(h)$ satisfies (1.1), and that (1.4) holds. If $V(x) \neq 0$ for $x \in \text{supp } u$, or if $g^{ij}$ is positive definite and $dV(x) \neq 0$ for $x \in \text{supp } u$, then

$$
\|u\|_{L^\infty} = O\left(h^{-\frac{n+1}{2}}\right), \quad n \geq 2.
$$

This result is the basis for Propositions 2.2 and 2.3 used in our proof. The case of Theorem 1.2 with $dV \neq 0$ is the most technically involved result of [2]. We do not know of any simpler way to obtain (1.2).

2. **Proof of Theorem 1.1**

By compactness of $K$, it suffices to prove uniform $L^\infty$ bounds on $u$ over a small ball about each point in $K$, where in our case the diameter of the ball can be taken to depend only on $C_N$ estimates for $g$ and $V$ over a unit sized neighborhood of $K$, for some large $N$. Without loss of generality we consider a ball centered at the origin in $\mathbb{R}^2$. Let

$$
B = \{x \in \mathbb{R}^2 : |x| < 1\}, \quad B^* = \{x \in \mathbb{R}^2 : |x| < 2\}.
$$

After a linear change of coordinates, we may assume that

$$
g^{ij}(0) = \delta^{ij}.
$$

Next, by replacing $V(x)$ by $cV(cx)$ and $g^{ij}(x)$ by $g^{ij}(cx)$, for some constant $c \leq 1$ depending on the $C^2$ norm of $g$ and $V$ over a unit neighborhood of $K$, we may assume that

$$
\sup_{x \in B^*} |V(x)| + |dV(x)| \leq 2, \quad \sup_{x \in B^*} |d^2V(x)| + \sum_{i,j=1}^2 |dg^{ij}(x)| \leq .01.
$$

This has the effect of multiplying $h$ by a constant in the equation (1.1), which can be absorbed into the constant $C_K$ in (1.2).

In general, we let

$$
C_N = \sup_{x \in B^*} \sup_{|\alpha| \leq N} \left( |\partial^\alpha V(x)| + \sum_{i,j=1}^2 |\partial^\alpha g^{ij}(x)| \right),
$$

and will deduce Theorem 1.1 as a corollary of the following

**Proposition 2.1.** Suppose $h \leq 1$, that $g, V$ satisfy (2.1) and (2.2), and that $u$ satisfies

$$
\| -h^2 \Delta_g u + Vu \|_{L^2(B^*)} \leq h, \quad \|u\|_{L^2(B^*)} \leq 1.
$$

Then

$$
\|u\|_{L^\infty(B^*)} \leq C h^{-\frac{1}{2}},
$$

where the constant $C$ depends only on $C_N$ in (2.3) for some fixed $N$. 
We start the proof of Proposition 2.1 by recording the following two propositions, which are consequences of Theorem 1.2.

**Proposition 2.2.** Suppose that (2.1)-(2.2) hold, and that \( \frac{1}{2} \leq |V(x)| \leq 2 \) for \(|x| \leq 2\). If the following holds, and \( h \leq 1 \),

\[
\| -h^2 \Delta_g u + Vu \|_{L^2(B^*)} \leq h, \quad \| u \|_{L^2(B^*)} \leq 1,
\]

then \( \| u \|_{L^\infty(B)} \leq C h^{-\frac{1}{2}} \), where \( C \) depends only on \( C_N \) in (2.3) for some fixed \( N \).

**Proposition 2.3.** Suppose that (2.1)-(2.2) hold, and that \( V(0) = 0 \) and \( |dV(0)| = 1 \). If the following holds, and \( h \leq 1 \),

\[
\| -h^2 \Delta_g u + Vu \|_{L^2(B^*)} \leq h, \quad \| u \|_{L^2(B^*)} \leq 1,
\]

then \( \| u \|_{L^\infty(B)} \leq C h^{-\frac{1}{2}} \), where \( C \) depends only on \( C_N \) in (2.3) for some fixed \( N \).

To see that these follow from Theorem 1.2, we first may assume that \( u \) is compactly supported in \(|x| < \frac{3}{2}\). Indeed, the assumptions imply \( \| du \|_{L^2(|x| < 3/2)} \lesssim h^{-1} \), so that one may cut off \( u \) by a smooth function which is supported in \(|x| < \frac{3}{2}\) and equals 1 for \(|x| < 1\) without affecting the hypotheses. We may then modify \( g \) and \( V \) outside \( B^* \) so that (2.2)-(2.3) are global bounds.

In Proposition 2.3 above, since \( |d^2 V| \leq .01 \), we have \( .98 \leq |dV(x)| \leq 1.02 \) for \(|x| \leq 2\), so since \( g \) is positive definite the conditions on \( g \) and \( V \) in Theorem 1.2 are met. We remark that the conditions of Proposition 2.3 guarantee that the zero set of \( V \) is a nearly-flat curve through the origin, although this is not strictly needed to apply the results of [2]. That the resulting constant \( C \) depends only on \( C_N \) for some fixed finite \( N \) follows from the proofs in [2].

Finally, the condition (1.4) that \( u - \chi^w(x, hD)u = O_{\varphi}(h^\infty) \) for some \( \chi \in C^\infty_c \) is not needed for Theorem 1.2 to hold for positive definite \( g^{ij} \) in dimension two. To see this, we note that if \( |V| < 2 \) and \( |g^{ij}(x) - \delta_{ij}| \leq .02 \) on the ball \(|x| < 2\), then if \( u \) is supported in \(|x| < \frac{3}{2}\) and \( \varphi(\xi) = 1 \) for \(|\xi| < 4\), condition (1.1) implies that

\[
\|(hD)^2(u - \varphi(hD)u)\|_{L^2} = O(h).
\]

This follows by the semiclassical pseudodifferential calculus (see [4, Theorem 4.29]), since for \( \varphi_0 \in C^\infty_c(\mathbb{R}^2) \) with \( \text{supp} \varphi_0 \subset B^* \), \( \varphi_0(x)(1 - \varphi(\xi))|\xi|^2/(|\xi|^2 + V(x)) \in S(\mathbb{R}^2 \times \mathbb{R}^2) \).
Hence, writing $\hat{u}(\xi)$ for the standard Fourier transform of $u$,

$$
\|u - \varphi(hD)u\|_{L^\infty} \leq \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} |1 - \varphi(h\xi)||\hat{u}(\xi)| \, d\xi
$$

(2.5)

\[
\leq C \int |h\xi|^2 |1 - \varphi(h\xi)||\hat{u}(\xi)|(1 + |h\xi|^2)^{-1} \, d\xi
\]

\[
\leq C \|(hD)^2(u - \varphi(hD)u)\|_{L^2} \left( \int_{\mathbb{R}^2} (1 + |h\xi|^2)^{-2} \, d\xi \right)^{\frac{1}{2}}
\]

\[
\leq Ch h^{-1} = C,
\]

an even better estimate than required. Hence we are reduced to proving estimates on $\varphi(hD)u$, which by compact support of $u$ satisfies (1.4).

We supplement Propositions 2.2 and 2.3 with the following two lemmas.

**Lemma 2.4.** Suppose that (2.1)-(2.2) hold, and that $|V(x)| \leq 99h$ for $|x| \leq 2h^{\frac{3}{2}}$. If the following holds, and $h \leq 1$,

$$
\left\| -h^2 \Delta_g u + V^\ast u \right\|_{L^2(|x| < 2h^{1/2})} \leq h , \quad \|u\|_{L^2(|x| < 2h^{1/2})} \leq 1 ,
$$

then $\|u\|_{L^\infty(|x| < h^{1/2})} \leq C h^{-\frac{1}{2}}$, where $C$ depends only on $C_N$ in (2.3) for some fixed $N$.

**Proof.** Consider the function $\tilde{u}(x) = h^{\frac{1}{2}}u(h^\frac{3}{2}x)$, and $\tilde{g}^{ij}(x) = g^{ij}(h^\frac{3}{2}x)$. Then, since $\|Vu\|_{L^2(|x| < 2h^{1/2})} \leq 99h$, we have

$$
\|\Delta_g \tilde{u}\|_{L^2(|x| < 2)} \leq 100 , \quad \|\tilde{u}\|_{L^2(|x| < 2)} \leq 1 .
$$

Since the spatial dimension equals 2, interior Sobolev estimates yield $\|\tilde{u}\|_{L^\infty(|x| < 1)} \leq C$, where we note that the conditions (2.1) and (2.2) hold for $\tilde{g}$ since $h^{\frac{3}{2}} \leq 1$. $\square$

**Lemma 2.5.** Suppose that (2.1)-(2.2) hold, and that $\frac{1}{2}c \leq |V(x)| \leq 2c$ for $|x| \leq 2c^{\frac{3}{2}}$. If the following holds, and $h \leq c \leq 1$,

$$
\left\| -h^2 \Delta_g u + V^\ast u \right\|_{L^2(|x| < 2c^{1/2})} \leq h , \quad \|u\|_{L^2(|x| < 2c^{1/2})} \leq 1 ,
$$

then $\|u\|_{L^\infty(|x| < c^{1/2})} \leq C h^{-\frac{1}{2}}$, where $C$ depends only $C_N$ in (2.3) for some fixed $N$.

**Proof.** Let $\tilde{u}(x) = c^{\frac{1}{2}}u(c^{\frac{3}{2}}x)$, $\tilde{g}^{ij}(x) = g^{ij}(c^{\frac{3}{2}}x)$, and $\tilde{V}(x) = c^{-1}V(c^{\frac{3}{2}}x)$. Note that the assumptions on $V(x)$ in the statement and in (2.2) imply that $|dV(x)| \leq c^{\frac{1}{2}}$ for $|x| < 2c^{1/2}$, so that $\tilde{V}$ satisfies (2.2), and the constants $C_N$ in (2.3) can only decrease for $c \leq 1$. Then with $\tilde{h} = c^{-1}h \leq 1$,

$$
\left\| -\tilde{h}^2 \Delta_g \tilde{u} + \tilde{V}\tilde{u} \right\|_{L^2(|x| < 2)} \leq \tilde{h} , \quad \|\tilde{u}\|_{L^2(|x| < 2)} \leq 1 .
$$

By Proposition 2.2, we have $\|\tilde{u}\|_{L^\infty(|x| < 1)} \leq C\tilde{h}^{-\frac{1}{2}}$, giving the desired result. $\square$
Proof of Proposition 2.1. It suffices to prove that for each \(|x_0| < 1\) there is some \(\frac{1}{2} \geq r > 0\) so that \(\|u\|_{L^\infty(|x-x_0|<r)} \leq C h^{-\frac{1}{2}}\), with a global constant \(C\). Without loss of generality we take \(x_0 = 0\).

We will split consideration up into four cases, depending on the relative size of \(|V(0)|\) and \(|dV(0)|\). Since for \(h\) bounded away from 0 the result follows by elliptic estimates, we will assume \(h \leq \frac{1}{4}\) so that \(h^{\frac{1}{2}}\) below is at most \(\frac{1}{2}\).

Case 1: \(|V(0)| \leq h\), \(|dV(0)| \leq 8h^{\frac{1}{2}}\). Since \(|d^2V(x)| \leq .01\), then Lemma 2.4 applies to give the result with \(r = h^{\frac{1}{2}}\).

Case 2: \(|V(0)| \leq h\), \(|dV(0)| \geq 8h^{\frac{1}{2}}\). Since we may add a constant of size \(h\) to \(V\) without affecting (2.4), we may assume \(V(0) = 0\). By rotating we may then assume

\[V(x) = \beta x_1 + f_{ij}(x)x_ix_j,\]

where \(\beta = |dV(0)| \geq 8h^{\frac{1}{2}}\). Dividing \(V\) by 4 if necessary we may assume \(\beta \leq \frac{1}{2}\). Let \(\tilde{u} = \beta u(\beta x), \tilde{g}^{ij}(x) = g^{ij}(\beta x),\) and

\[\tilde{V}(x) = \beta^{-2}V(\beta x) = x_1 + f_{ij}(\beta x)x_ix_j.\]

With \(\tilde{h} = \beta^{-2}h < 1\) we have

\[\| - \tilde{h}^2 \Delta \tilde{g} \tilde{u} + \tilde{V} \tilde{u} \|_{L^2(|x|<2)} \leq \tilde{h}, \quad \|\tilde{u}\|_{L^2(|x|<2)} \leq 1.\]

Proposition 2.3 applies, since \(\tilde{g}\) and \(\tilde{V}\) satisfy (2.1)-(2.2), and the constants \(C_N\) in (2.3) for \(\tilde{g}\) and \(\tilde{V}\) are bounded by those for \(g\) and \(V\). Thus \(\|\tilde{u}\|_{L^\infty(|x|<1)} \leq C\tilde{h}^{-\frac{1}{2}}\), giving the desired result on \(u\) with \(r = |dV(0)|\).

Case 3: \(|V(0)| \geq h\), \(|dV(0)| \leq 9|V(0)|^{\frac{1}{2}}\). In this case, with \(c = |V(0)|\), it follows that \(\frac{1}{2}c \leq |V(x)| \leq 2c\) for \(|x| \leq \frac{1}{20}c^{\frac{1}{2}}\). We may apply Lemma 2.5 with \(V\) replaced by \(\frac{1}{1600}V\) to get the desired result with \(r = \frac{1}{40}|V(0)|^{\frac{1}{2}}\).

Case 4: \(|V(0)| \geq h\), \(|dV(0)| \geq 9|V(0)|^{\frac{1}{2}}\). Since \(|d^2V(x)| \leq .01\), it follows that there is a point \(x_0\) with \(|x_0| \leq \frac{1}{8}|V(0)|^{\frac{1}{2}}\) where \(V(x_0) = 0\). Since \(|dV(x_0)| \geq 8|V(0)|^{\frac{1}{2}} \geq 8h^{\frac{1}{2}}\), we may translate and apply Case 2 to get \(L^\infty\) bounds on \(u\) over a neighborhood of radius \(|dV(x_0)|\) about \(x_0\). This neighborhood contains the neighborhood about 0 of radius \(r = .9998|dV(0)|\).

\[\square\]

3. A COUNTER-EXAMPLE FOR INDEFINITE \(g\).

In [2, Section 5], it was shown that there exist \(u_h\) for which

\[| - h^2(\partial_{x_1}^2 - \partial_{x_2}^2)u_h + (x_1^2 - x_2^2)u_h \|_{L^2} \leq h, \quad \|u_h\|_{L^2} \leq 1,\]

for which \(\|u_h\|_{L^\infty} \approx |\log h|^\frac{1}{2}h^{-\frac{1}{2}}\), showing that the assumption of definiteness of \(g\) cannot be relaxed to non-degeneracy in the main theorem. In [2, Theorem 6] the positive result
was established showing that this growth of \( \|u_h\|_{L^\infty} \) for indefinite, non-degenerate \( g \) in two dimensions is in fact worst case.

The example of [2] was produced using harmonic oscillator eigenstates. Here we present a different construction of such a \( u_h \) with similar \( L^\infty \) growth to help illustrate the role played by the degeneracy of \( g \). The idea is to produce a collection \( u_{h,j} \) of functions satisfying (3.1) (or equivalent), for which \( u_{h,j}(0) = h^{-\frac{1}{2}} \), and where \( j \) runs over \( \approx |\log h| \) different values. The examples will have disjoint frequency support, hence are orthogonal in \( L^2 \).

Upon summation over \( j \) the \( L^2 \) norm then grows as \( |\log h|^{\frac{1}{2}} \), whereas the \( L^\infty \) norm grows as \( |\log h|h^{-\frac{1}{2}} \), yielding an example with worst case growth after normalization.

We start by considering the form \( \xi_1 \xi_2 \) with \( V = 0 \). To assure that \( \| h^2 \partial_{x_1} \partial_{x_2} u_h \|_{L^2} \leq h \), we will take the Fourier transform of \( u_h \) to be contained in the set \( |\xi_1 \xi_2| \leq 2h^{-1} \), as well as \( |\xi| \leq 2h^{-1} \) to satisfy the frequency localization condition [2, (1.4)]. Our example is then based on the fact that one can find \( \approx |\log h| \) disjoint rectangles, each of volume \( h^{-1} \), within this region, as illustrated in the diagram. Each \( u_{h,j} \) will be an appropriately scaled Schwartz function with Fourier transform localized to one of the rectangles.

We now fix \( \psi, \chi \in C_c^\infty(\mathbb{R}) \), with \( 0 \leq \psi(x) \leq 2 \) and \( 0 \leq \chi(x) \leq 1 \), with \( \int \psi = \int \chi = 1 \), and where

\[
\text{supp } \psi \subset [1, 2], \quad \text{supp } \chi \subset [-1, 1].
\]

We additionally assume \( \chi(0) = 1 \).

Let

\[
u_{h,j}(x) = h^{\frac{1}{2}} \int e^{ix_1 \xi_1 + ix_2 \xi_2} \chi(2^j h \xi_1) \psi(2^{-j} \xi_2) d\xi_1 d\xi_2 = h^{-\frac{1}{2}} \chi(2^j h^{-1} x_1) \psi(2^j x_2).
\]

By the Plancherel theorem, \( \|u_{h,j}\|_{L^2} \approx 1 \) and \( \|h^2 D_1 D_2 u_{h,j}\|_{L^2} \lesssim h \). Furthermore, \( u_{h,j}(0) = h^{-\frac{1}{2}} \). By disjointness of the Fourier transforms, for \( i \neq j \) we have \( \langle u_{h,i}, u_{h,j} \rangle = 0 \), and similarly \( \langle \partial_{x_1} \partial_{x_2} u_{h,i}, \partial_{x_1} \partial_{x_2} u_{h,j} \rangle = 0 \).

We then form

\[
u_h(x) = |\log h|^{-\frac{1}{2}} \sum_{1 \leq 2^j \leq h^{-1}} u_{h,j}(x).
\]
Since there are $\approx |\log h|$ terms in the sum, and the terms are orthogonal in $L^2$, it follows that
\[ \|u_h\|_{L^2} \approx 1, \quad \|h^2 \partial_{x_1} \partial_{x_2} u_h\|_{L^2(\mathbb{R}^2)} \lesssim h, \quad u_h(0) \approx |\log h|^{1/2} h^{-1/2}. \]
Although the example is not compactly supported, it is rapidly decreasing (uniformly so for $h < 1$), and one may smoothly cutoff to a bounded set without changing the estimates.

We observe that for this example it also holds that
\[ \|x_1 x_2 u_h\|_{L^2} \lesssim h. \]
Hence, $u_h$ is also a counterexample for the form $\xi_1 \xi_2 \pm x_1 x_2$. Rotating by $\pi/4$ gives the form $\xi_1^2 - \xi_2^2 \pm (x_1^2 - x_2^2)$, including in particular the form considered in [2, Section 6].

We also observe that $x_1^2 u_h$ will be $O_{L^2}(h)$ if one restricts the sum in $u_h$ to $1 \leq 2^j \leq h^{-1/2}$, which still has $\approx |\log h|$ values of $j$, and thus exhibits the same $L^\infty$ growth as $u_h$. This idea does not however work to yield a counterexample for the form $\xi_1 \xi_2 + x_1^2 + x_2^2$.

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