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SHARP $L^2 \to L^q$ BOUNDS ON SPECTRAL PROJECTORS FOR LOW REGULARITY METRICS

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Abstract. We establish $L^2 \to L^q$ mapping bounds for unit-width spectral projectors associated to elliptic operators with C^s coefficients, in the case $1 \le s \le 2$. Examples of Smith-Sogge [6] show that these bounds are best possible for q less than the critical index. We also show that L^∞ bounds hold with the same exponent as in the case of smooth coefficients.

1. Introduction

The goal of this paper is to study the L^p norms of eigenfunctions, and approximate eigenfunctions, of elliptic second order differential operators with low regularity coefficients, on compact manifolds without boundary. We consider the eigenvalues $-\lambda^2$ and eigenfunctions ϕ for an equation

$$d^*(a\,d\phi) + \lambda^2 \rho\,\phi = 0\,. \tag{1}$$

Here we assume $\rho > 0$ is a real, positive measurable function, and $a_x : T^*_x(M) \to T_x(M)$ is the transformation associated to a real symmetric form on $T^*_x(M)$, also strictly positive and measurable in x. The manifold M and volume form dx are assumed smooth, and d^* is the transpose of the differential d with respect to dx. This setting includes the most general elliptic second order operator on M, assumed self-adjoint with respect to some measurable volume form $\rho\,dx$, and assumed to annihilate constants, and hence of the form $\rho^{-1}d^*ad$. For limited regularity a and ρ we pose the problem as above to avoid domain considerations.

If we consider the real quadratic forms

$$Q_0(f,g) = \int_M f\,g\,\rho\,dx\,, \qquad Q_1(f,g) = Q_0(f,g) + \int_M a(df,dg)\,dx\,,$$

then

$$Q_0(f,f) = \|f\|^2_{L^2(M,\rho dx)}\,, \qquad Q_1(f,f) \approx \|f\|^2_{H^1(M)}\,,$$

hence Q_0 is compact relative to Q_1 by Rellich's lemma. By the standard argument of simultaneously diagonalizing Q_0 and Q_1, there exists a complete orthonormal basis ϕ_j for $L^2(M,\rho\,dx)$ consisting of eigenfunctions for (1), with $\lambda_j \to \infty$.

The object of this paper is to establish bounds on the $L^2 \to L^q$ operator norm of the unit-width spectral projectors for (1). Let Π_λ be the projection of $L^2(M,\rho\,dx)$ onto the subspace spanned by the eigenfunctions of (1) for which $\lambda_j \in [\lambda, \lambda+1]$. In

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the case that the coefficients ρ and a are C^∞, the following estimates hold, and are best possible in terms of the exponent of λ,

$$\big\|\Pi_\lambda f\big\|_{L^q(M)} \le C\,\lambda^{\frac{n-1}{2}(\frac12-\frac1q)}\,\|f\|_{L^2(M)}\,, \qquad 2\le q\le q_n\,, \tag{2}$$

$$\big\|\Pi_\lambda f\big\|_{L^q(M)} \le C\,\lambda^{n(\frac12-\frac1q)-\frac12}\,\|f\|_{L^2(M)}\,, \qquad q_n\le q\le \infty\,, \tag{3}$$

where

$$q_n = \frac{2(n+1)}{n-1}$$

For C^∞ metrics the estimates at $q = q_n$ are due to Sogge [8]. The estimate for $q=\infty$ is related to the spectral counting remainder estimates of Avakumović-Levitan-Hörmander; it can also be obtained from Sogge's estimate by Sobolev embedding. The case $q = 2$ is of course trivial, and all other values of q follow from these endpoints by interpolation.

In [5], both estimates (2) and (3) were established on the full range of q for the case that both a and ρ are of class $C^{1,1}$.

On the other hand, Smith-Sogge [6] and Smith-Tataru [7] constructed examples, for each $0 < s < 2$, of functions a and ρ with coefficients of class C^s (Lipschitz in case $s = 1$) for which there exist eigenfunctions ϕ_λ such that for all $q \ge 2$

$$\|\phi_\lambda\|_{L^q(M)} \ge C\,\lambda^{\frac{n-1}{2}(\frac12-\frac1q)(1+\sigma)}\|\phi_\lambda\|_{L^2(M)}\,,$$

where $C > 0$ is independent of λ, and where

$$\sigma = \frac{2-s}{2+s}$$

For $2 < q < \frac{2(n+2s^{-1})}{n-1}$, this shows that the spectral projection estimates for C^s metrics with $s < 2$ can be strictly worse than in the C^2 case.

In this paper, we consider the case of coefficients a and ρ of class C^s for $1 \le s < 2$ (Lipschitz in case $s = 1$.) We start by establishing the following bound, which by the examples of [6] is best possible on the indicated range of q.

Theorem 1. *Assume that the coefficients a and ρ are either of class C^s for some $1 < s < 2$, or Lipschitz class if $s = 1$. Let Π_λ denote the L^2-projection onto the subspace spanned by eigenfunctions of* (1) *with $\lambda_j \in [\lambda, \lambda+1]$. Then*

$$\big\|\Pi_\lambda f\big\|_{L^q(M)} \le C\,\lambda^{\frac{n-1}{2}(\frac12-\frac1q)(1+\sigma)}\,\|f\|_{L^2(M)}\,, \qquad 2\le q\le q_n\,.$$

Applying Sobolev embedding to the estimate at $q = q_n$ would not yield the correct bound for $q = \infty$. However, the proof of Theorem 1 also yields no-loss estimates on small sets. Precisely, we will establish the following local estimate, with constant uniform over the balls B.

Theorem 2. *Let $B_R \subset M$ be a ball of radius $R = \lambda^{-\sigma}$. Then under the same conditions as Theorem 1*

$$\big\|\Pi_\lambda f\big\|_{L^q(B_R)} \le C\,\lambda^{n(\frac12-\frac1q)-\frac12}\,\|f\|_{L^2(M)}\,, \qquad q_n\le q\le \infty\,. \tag{4}$$

Interpolating with the trivial L^2 estimate establishes the estimate (2) on such balls B_R. Since the constant C in (4) is uniform for all balls B_R, we obtain the same global $L^2 \to L^\infty$ mapping properties in the case of Lipschitz coefficients as in the case of smooth coefficients,

$$\|\Pi_\lambda f\|_{L^\infty(M)} \le C\, \lambda^{\frac{n-1}{2}}\, \|f\|_{L^2(M)}\,. \tag{5}$$

A corollary of this result is the Hörmander multiplier theorem on compact manifolds for functions of elliptic operators with Lipschitz coefficients, as shown by results of Duong-Ouhabaz-Sikora [1], section 7.2. We note that, in related work, Ivrii [2] has obtained the sharp spectral counting remainder estimate for operators with coefficients of regularity slightly stronger than Lipschitz.

The proof of Theorem 2 that we will present requires that q be not too large, but in all dimensions works for $q = q_n$. We therefore show here how heat kernel estimates permit us to deduce (4) for all $q \ge q_n$ from the case $q = q_n$. For this, let H_λ denote the heat kernel at time $\lambda^{-2} \le 1$ for the diffusion system associated to (1). By Theorem 6.3 of Saloff-Coste [4], the integral kernel h_λ of H_λ satisfies

$$|h_\lambda(x,y)| \le C\, \lambda^n \exp(-c\, \lambda^2 d(x,y)^2)\,.$$

By Young's inequality, then for $q_n \le q \le \infty$

$$\begin{aligned}\|\Pi_\lambda f\|_{L^q(B_R)} &\le C\, \lambda^{n(\frac{1}{q_n}-\frac{1}{q})} \|H_\lambda^{-1}\Pi_\lambda f\|_{L^{q_n}(B_R^*)} + C_N\, \lambda^{-N} \|H_\lambda^{-1}\Pi_\lambda f\|_{L^2(M\setminus B_R^*)}\\ &\le C\, \lambda^{n(\frac12-\frac1q)-\frac12} \|f\|_{L^2(M)}\end{aligned}$$

where we use (4) at $q = q_n$ with B_R replaced by its double B_R^*, and the fact that $\|H_\lambda^{-1}\Pi_\lambda f\|_{L^2} \approx \|\Pi_\lambda f\|_{L^2}$ since $\exp(\lambda_j^2/\lambda^2) \approx 1$ for $\lambda_j \in [\lambda, \lambda+1]$.

If we interpolate the estimate of Theorem 1 at $q = q_n$ with the estimate (5), then we obtain the following.

Corollary 3. *Under the same conditions as Theorem 1*

$$\big\|\Pi_\lambda f\big\|_{L^q(M)} \le C\, \lambda^{n(\frac12-\frac1q)-\frac12+\frac{\sigma}{q}}\, \|f\|_{L^2(M)}\,, \qquad q_n \le q \le \infty\,.$$

For $q_n < q < \infty$, however, the exponent is strictly larger than that predicted by the examples of [6]. It is not currently known what the sharp exponent is for this range.

The key idea in our proof is that a C^s function is well approximated on sets of diameter $R = \lambda^{-\sigma}$ by a C^2 function, up to an error which is suitably bounded when dealing with eigenfunctions localized to frequency λ. In effect, rescaling by R reduces matters to a C^2 situation, where no-loss estimates hold. The loss of $\lambda^{\frac{\sigma}{q}}$ comes from adding up the bounds over $\approx R^{-1}$ disjoint sets.

This scaling parameter R occurs in the examples of Smith-Sogge [6] and Smith-Tataru [7]. The idea of scaling by R to prove L^p estimates was first used by Tataru in [9], to establish Strichartz-type estimates for time-dependent wave equations with C^s coefficients, yielding improved existence theorems for a class of quasilinear hyperbolic equations.

Notation. By a C^s function on $\mathbb{R}^n$, for $1 < s \le 2$ we understand a continuously differentiable function f such that

$$\|f\|_{C^s} = \|f\|_{L^\infty(\mathbb{R}^n)} + \|df\|_{L^\infty(\mathbb{R}^n)} + \sup_{h\in\mathbb{R}^n} |h|^{1-s}\, \|df(\cdot+h) - df(\cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty\,.$$

Thus, C^s coincides with $C^{1,s-1}$ for $s \in (1,2]$. For $s = 1$, we use C^1 to mean Lipschitz. For $0 < s < 1$ we take C^s to be the standard Holder class.

We use d to denote the differential taking functions to covector fields, and d^* its adjoint with respect to dx. When working on $\mathbb{R}^n$, $d = (\partial_1, \dots, \partial_n)$, and d^* is the standard divergence operator.

The notation $A \lesssim B$ means $A \le C\,B$, where C is a constant that depends only on the C^s norm of a and ρ, as well as on universally fixed quantities, such as the manifold M and the non-degeneracy of a and ρ. In particular, C can be taken to depend continuously on a and ρ in the C^s norm, so our estimates are uniform under small C^s perturbations of a and ρ.

2. Scaling Arguments

Our starting point is the following square-function estimate for solutions to the Cauchy problem. For C^∞ coefficients this was established by Mockenhaupt-Seeger-Sogge [3]. The version we need for $C^{1,1}$ metrics is Theorem 1.3 of [5]. That theorem was stated under the condition $F = 0$ and for coefficients which are constant for large x, but these conditions are easily dropped by the Duhamel principle and a partition of unity argument.

Theorem 4. *Suppose that a and ρ are defined globally on $\mathbb{R}^n$, and that*

$$\|a^{ij} - \delta^{ij}\|_{C^{1,1}(\mathbb{R}^n)} + \|\rho - 1\|_{C^{1,1}(\mathbb{R}^n)} \le c_0\,,$$

where c_0 is a small constant depending only on n. Let u solve the Cauchy problem

$$\rho(x)\,\partial_t^2 u(t,x) - d^*\big(a(x)\,du(t,x)\big) = F(t,x)\,, \quad u(0,x) = u_0(x)\,, \quad \partial_t u(0,x) = u_1(x)\,.$$

Then

$$\|u\|_{L^{q_n}_x L^2_t(\mathbb{R}^n \times [-1,1])} \lesssim \|u_0\|_{H^{\frac{1}{q_n}}} + \|u_1\|_{H^{\frac{1}{q_n}-1}} + \|F\|_{L^1_t H^{\frac{1}{q_n}-1}} \tag{6}$$

We first deduce the following corollary which is more useful for our purposes.

Corollary 5. *Suppose that f satisfies an equation on $\mathbb{R}^n$ of the form*

$$d^*(a\,df) + \mu^2 \rho\, f = d^* g_1 + g_2\,.$$

If a and ρ satisfy the condition of Theorem 4, then

$$\|f\|_{L^{q_n}} \lesssim \mu^{\frac{1}{q_n}} \big(\,\|f\|_{L^2} + \mu^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1}\|g_2\|_{L^2}\big)\,. \tag{7}$$

Proof. Let $S_r = S_r(D)$ denote a smooth cutoff on the Fourier transform side to frequencies of size $|\xi| \le r$. Let $a_\mu = S_{c^2\mu} a$, for c to be chosen suitably small. Then

$$\|(a - a_\mu) df\|_{L^2} \lesssim c^{-2}\mu^{-1}\|df\|_{L^2}\,, \qquad \mu^2\|(\rho - \rho_\mu) f\|_{L^2} \lesssim c^{-2}\mu\,\|f\|_{L^2}\,,$$

and thus we may replace a and ρ by a_μ and ρ_μ at the expense of absorbing the above two terms into g_1 and g_2, which does not change the size of the right hand side of (7).

Next, let $f_{<\mu} = S_{c\mu} f$. Since

$$\big\|[S_{c\mu}, a_\mu]\big\|_{L^2 \to L^2} \lesssim (c\mu)^{-1}\,,$$

and similarly for $[S_{c\mu}, \rho_\mu]$, we can absorb the commutator terms into g_1 and g_2, and since all terms are localized to frequencies less than μ we can write

$$d^*(a_\mu\, df_{<\mu}) + \mu^2 \rho_\mu\, f_{<\mu} = g_{<\mu}\,, \tag{8}$$

where

$$\|g_{<\mu}\|_{L^2} \lesssim \mu\,\|f\|_{L^2} + \|df\|_{L^2} + \mu\,\|g_1\|_{L^2} + \|g_2\|_{L^2}$$

Since $\|d^*(a_\mu df_{<\mu})\|_{L^2} \lesssim (c\mu)^2\|f_{<\mu}\|_{L^2}$, for c suitably small the L^2 norm of the left hand side of (8) is comparable to $\mu^2\|f_{<\mu}\|_{L^2}$, hence we have

$$\|f_{<\mu}\|_{L^2} \lesssim \mu^{-1}\big(\,\|f\|_{L^2} + \mu^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1}\|g_2\|_{L^2}\big)$$

Sobolev embedding now implies (7) if f is replaced on the left hand side by $f_{<\mu}$. In fact there is a gain of $\mu^{-\frac{1}{2}}$, since $\frac{1}{q_n} = n(\frac{1}{2} - \frac{1}{q_n}) - \frac{1}{2}$.

If we let $f_{>\mu} = f - S_{c^{-1}\mu}f$, then similar arguments let us write

$$d^*(a_\mu\, df_{>\mu}) + \mu^2\rho_\mu\, f_{>\mu} = d^*g_{>\mu} \tag{9}$$

where now $g_{>\mu}$, like $f_{>\mu}$, is frequency localized to frequencies larger than $c^{-1}\mu$, and

$$\|g_{>\mu}\|_{L^2} \lesssim \|f\|_{L^2} + \mu^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1}\|g_2\|_{L^2}$$

Taking the inner product of both sides of (9) against $f_{>\mu}$ yields

$$\|df_{>\mu}\|_{L^2}^2 - 4\mu^2\|f_{>\mu}\|_{L^2}^2 \lesssim \|g_{>\mu}\|_{L^2}\|df_{>\mu}\|_{L^2}$$

and by the frequency localization of $f_{>\mu}$ we obtain

$$\|f_{>\mu}\|_{H^1} \lesssim \|f\|_{L^2} + \mu^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1}\|g_2\|_{L^2}$$

Since $n(\frac{1}{2} - \frac{1}{q_n}) = \frac{1}{q_n} + \frac{1}{2} \le 1$, Sobolev embedding yields (7) if f is replaced on the left hand side by $f_{>\mu}$. As above, there is in fact a gain of $\mu^{-\frac{1}{2}}$ for this term.

We now let $f_\mu = S_{c^{-1}\mu}f - S_{c\mu}f$, and as above write

$$d^*(a_\mu\, df_\mu) + \mu^2\rho_\mu\, f_\mu = g_\mu$$

where now f_μ and g_μ are localized to frequencies comparable to μ, and

$$\|g_\mu\|_{L^2} \lesssim \mu\,\|f\|_{L^2} + \|df\|_{L^2} + \mu\,\|g_1\|_{L^2} + \|g_2\|_{L^2}$$

Setting $u(t,x) = \cos(\mu t)f_\mu(x)$, we apply (6) to deduce

$$\|f_\mu\|_{L^q} \lesssim \mu^{\frac{1}{q_n}}\big(\,\|f_\mu\|_{L^2} + \mu^{-1}\|g_\mu\|_{L^2}\big)$$

which yields (7) for this term. $\square$

Remark. For future use, we note that in the proof of Corollary 5 the assumption that $a \in C^{1,1}$ was used only at the last step, in order to deduce that (6) holds. The commutator and approximation bounds require only that a and ρ be Lipschitz. In particular, the bounds on $f_{<\mu}$ and $f_{>\mu}$ hold for Lipschitz a and ρ.

Corollary 6. *Let Q be a unit cube and Q^* its double. Suppose that a and ρ are bounded and measurable, and that there exist $C^{1,1}$ functions $\tilde{a}$ and $\tilde{\rho}$ satisfying the conditions of Theorem 4 such that*

$$\|a - \tilde{a}\|_{L^\infty(Q^*)} + \|\rho - \tilde{\rho}\|_{L^\infty(Q^*)} \le \mu^{-1}$$

Suppose that on Q^* we have

$$d^*(a\,df) + \mu^2\rho\, f = d^*g_1 + g_2$$

Then

$$\|f\|_{L^{q_n}(Q)} \lesssim \mu^{\frac{1}{q_n}}\big(\,\|f\|_{L^2(Q^*)} + \mu^{-1}\|df\|_{L^2(Q^*)} + \|g_1\|_{L^2(Q^*)} + \mu^{-1}\|g_2\|_{L^2(Q^*)}\big)$$

The constant in the inequality is uniform for $\mu \ge 1$.

Proof. Let ϕ be a smooth function, equal to 1 on Q and supported in Q^*. Then

$$\begin{aligned} d^*(a\,d(\phi f)) + \mu^2\rho\,(\phi f) &= d^*\big[\,(a\,d\phi)f + \phi g_1\big] + \big[(a\,d\phi)\cdot df - (d\phi)\cdot g_1 + \phi g_2\big] \\ &= d^*\tilde g_1 + \tilde g_2 \end{aligned}$$

where for $\mu \ge 1$

$$\|\tilde g_1\|_{L^2} + \mu^{-1}\|\tilde g_2\|_{L^2} \lesssim \|f\|_{L^2(Q^*)} + \mu^{-1}\|df\|_{L^2(Q^*)} + \|g_1\|_{L^2(Q^*)} + \mu^{-1}\|g_2\|_{L^2(Q^*)}$$

One may similarly absorb $(a-\tilde a)d(\phi f)$ into $\tilde g_1$, and $\mu^2(\rho-\tilde\rho)(\phi f)$ into $\tilde g_2$. The result now follows from (7). □

Corollary 7. *Suppose that a and ρ are of class C^s, with $0 \le s \le 2$, and that*

$$\|a^{ij} - \delta^{ij}\|_{C^s(\mathbb{R}^n)} + \|\rho - 1\|_{C^s(\mathbb{R}^n)} \le c_0\,,$$

where c_0 is a small constant depending only on n.

Suppose that $R = \lambda^{-\sigma}$, where $\sigma = \frac{2-s}{2+s}$ and $\lambda \ge 1$. Assume Q_R is a cube of sidelength R, Q_R^* is its double, and on Q_R^* the following equation holds

$$d^*(a\,df) + \lambda^2\rho\,f = d^*g_1 + g_2$$

Then

$$\begin{aligned} \|f\|_{L^{q_n}(Q_R)} \lesssim R^{-\frac12}\lambda^{\frac{1}{q_n}}\big(\,&\|f\|_{L^2(Q_R^*)} + \lambda^{-1}\|df\|_{L^2(Q_R^*)} \\ &+ R\,\|g_1\|_{L^2(Q_R^*)} + R\lambda^{-1}\|g_2\|_{L^2(Q_R^*)}\big)\,. \end{aligned}$$

Proof. We use the notation $f_R(x) = f(Rx)$. Then, for $\mu = R\lambda = \lambda^{1-\sigma}$,

$$d^*(a_R\,df_R) + \mu^2\rho_R\,f_R = R\,d^*g_{1,R} + R^2 g_{2,R}$$

holds on Q^*, with Q a unit cube. If $\tilde a = S_{\mu^{1/2}}a_R$, then

$$\|\tilde a - a_R\|_{L^\infty} \lesssim \mu^{-\frac12 s}R^s\|a - I\|_{C^s} = c_0\mu^{-1}$$

By the frequency localization, $\tilde a$ satisfies the conditions of Theorem 4. We may thus apply Corollary 6 to yield

$$\begin{aligned} \|f_R\|_{L^{q_n}(Q)} \lesssim (R\lambda)^{\frac{1}{q_n}}\big(\,&\|f_R\|_{L^2(Q^*)} + \lambda^{-1}\|(df)_R\|_{L^2(Q^*)} \\ &+ R\,\|g_{1,R}\|_{L^2(Q^*)} + R\lambda^{-1}\|g_{2,R}\|_{L^2(Q^*)}\big) \end{aligned}$$

Recalling that $\frac{1}{q_n} = n(\frac12 - \frac{1}{q_n}) - \frac12$, this yields the corollary after rescaling. □

3. Proof of Theorem 1

The proof of Corollary 7 works for all $s \in [0,2]$, but the energy estimates of this section require that a and ρ be Lipschitz, hence we assume $s \ge 1$ for the remainder.

The projection $\Pi_\lambda f$ satisfies

$$\begin{aligned} \|d^*\big(a\,d\,(\Pi_\lambda f)\big) + \lambda^2\rho\,\Pi_\lambda f\|_{L^2(M,\rho dx)} &\le (2\lambda+1)\,\|\Pi_\lambda f\|_{L^2(M,\rho dx)} \\ \|d\,\Pi_\lambda f\|_{L^2(M,\rho dx)} &\lesssim (\lambda+1)\,\|\Pi_\lambda f\|_{L^2(M,\rho dx)} \end{aligned}$$

hence Theorem 1 follows from showing that, if the following holds on M

$$d^*(a\,df) + \lambda^2\rho\,f = g \tag{10}$$

then uniformly for $\lambda \geq 1$

$$\|f\|_{L^{q_n}(M)} \lesssim \lambda^{\frac{1+\sigma}{q_n}} \left(\|f\|_{L^2(M)} + \lambda^{-1} \|df\|_{L^2(M)} + \lambda^{-1} \|g\|_{L^2(M)} \right) \tag{11}$$

Assume that (10) holds, and let ϕ be a C^2 bump function on M. Then

$$d^*(a\, d(\phi f)) + \lambda^2 \rho\, \phi f = f\, d^*(a\, d\phi) + \langle a\, d\phi, df \rangle + \phi g$$

Absorbing the terms on the right into g leaves the right hand side of (11) unchanged, hence by a partition of unity argument we may assume that f is supported in a suitably small coordinate neighborhood on M.

We choose coordinate patches so that, in local coordinates, the conditions of Corollary 7 are satisfied after extending a and ρ to all of $\mathbb{R}^n$. Thus, we have an equation of the form (10) on $\mathbb{R}^n$, with f and g supported in a unit cube.

We next decompose $f = f_{<\lambda} + f_{>\lambda} + f_\lambda$ as in the proof of Corollary 5. As remarked following that proof, the bounds on $f_{<\lambda}$ and $f_{>\lambda}$ hold for a and ρ Lipschitz, hence we are reduced to considering f_λ, for which we have an equation

$$d^*(a_\lambda\, df_\lambda) + \lambda^2 \rho_\lambda\, f_\lambda = g_\lambda$$

where a_λ and ρ_λ are localized to frequencies smaller than $c^2\lambda$, and both f_λ and g_λ are localized to frequencies of size comparable to λ.

We then decompose $f_\lambda = \sum_{j=1}^N \Gamma_j f_\lambda$, where each $\Gamma_j = \Gamma_j(D)$ is an order 0 multiplier, with symbol $\Gamma_j(\xi)$ supported where $|\xi| \approx \lambda$ and in a cone of suitably small angle. It then suffices to bound each $\|\Gamma_j f_\lambda\|_{L^{q_n}(Q)}$ by the right hand side of (11). Without loss of generality we consider a term with $\Gamma(\xi)$ localized to a small cone about the ξ_1 axis.

We write

$$d^*(a_\lambda\, d\,\Gamma f_\lambda) + \lambda^2 \rho_\lambda\, \Gamma f_\lambda = \Gamma g_\lambda + d^*[a_\lambda, \Gamma]\, df_\lambda + \lambda^2 [\rho_\lambda, \Gamma]\, f_\lambda$$

Simple commutator estimates show that the right hand side has L^2 norm bounded by $\lambda \|f\|_{L^2} + \|g\|_{L^2}$, hence we are reduced to establishing

$$\|f\|_{L^{q_n}(Q)} \lesssim \lambda^{\frac{1+\sigma}{q_n}} \left(\|f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|df\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|g\|_{L^2(\mathbb{R}^n)} \right) \tag{12}$$

for f satisfying the equation

$$d^*(a_\lambda\, df) + \lambda^2 \rho_\lambda\, f = g$$

where $\widehat{f}(\xi)$ and $\widehat{g}(\xi)$ are localized to $|\xi| \approx \lambda$ and ξ in a small cone about the ξ_1 axis.

By Corollary 7, for any cube Q_R of sidelength $R = \lambda^{-\sigma}$, we have

$$\|f\|_{L^{q_n}(Q_R)} \lesssim \lambda^{\frac{1}{q_n}} \left(R^{-\frac12} \|f\|_{L^2(Q_R^*)} + R^{-\frac12} \lambda^{-1} \|df\|_{L^2(Q_R^*)} + R^{\frac12} \lambda^{-1} \|g\|_{L^2(Q_R^*)} \right). \tag{13}$$

Let S_R denote a slab of the form $\{x \in \mathbb{R}^n : |x_1 - c| \leq R\}$. By summing over cubes Q_R contained in S_R, and noting $R \leq 1$, we obtain

$$\|f\|_{L^{q_n}(S_R)} \lesssim \lambda^{\frac{1}{q_n}} \Big(R^{-\frac12} \|f\|_{L^2(S_R^*)} + R^{-\frac12} \lambda^{-1} \|df\|_{L^2(S_R^*)} + \lambda^{-1} \|g\|_{L^2(S_R^*)} \Big) \tag{14}$$

We will show that

$$R^{-\frac12} \left(\|f\|_{L^2(S_R^*)} + \lambda^{-1} \|df\|_{L^2(S_R^*)} \right) \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|df\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|g\|_{L^2(\mathbb{R}^n)} \tag{15}$$

Given this, inequality (12) follows from (14) by adding over the $R^{-1} = \lambda^\sigma$ disjoint slabs that intersect Q. Also, the bound (13) implies the conclusion of Theorem 2 for $q = q_n$ (hence for all q by the heat kernel arguments following that theorem.)

We establish (15) by energy inequality arguments. Let V denote the vector field

$$V = 2(\partial_1 f)\, a_\lambda\, df + \big(\lambda^2 \rho_\lambda\, f^2 - \langle a_\lambda\, df, df\rangle\big)\overrightarrow{e_1}$$

Then

$$d^*V = 2(\partial_1 f)\, g + \lambda^2(\partial_1\rho_\lambda) f^2 - \langle(\partial_1 a_\lambda) df, df\rangle$$

Applying the divergence theorem on the set $x_1 \le r$ yields

$$\int_{x_1=r} V_1\, dx' \lesssim \lambda^2 \|f\|^2_{L^2(\mathbb{R}^n)} + \|df\|^2_{L^2(\mathbb{R}^n)} + \|g\|^2_{L^2(\mathbb{R}^n)}$$

Since a_λ and ρ are pointwise close to the flat metric, we have pointwise that

$$V_1 \ge \tfrac{3}{4}|\partial_1 f|^2 + \tfrac{3}{4}\lambda^2|f|^2 - |\partial_{x'} f|^2$$

The frequency localization of $\widehat{f}$ to $|\xi'| \le c\lambda$ yields

$$\int_{x_1=r} V_1\, dx' \ge \frac{1}{2}\int_{x_1=r} |df|^2 + \lambda^2|f|^2\, dx'$$

Integrating this over r in an interval of size R yields (15). $\square$

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