SHARP $L^2 \to L^q$ BOUNDS ON SPECTRAL PROJECTORS FOR LOW REGULARITY METRICS

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Abstract. We establish $L^2 \to L^q$ mapping bounds for unit-width spectral projectors associated to elliptic operators with $C^s$ coefficients, in the case $1 \leq s \leq 2$. Examples of Smith-Sogge [6] show that these bounds are best possible for $q$ less than the critical index. We also show that $L^\infty$ bounds hold with the same exponent as in the case of smooth coefficients.

1. Introduction

The goal of this paper is to study the $L^p$ norms of eigenfunctions, and approximate eigenfunctions, of elliptic second order differential operators with low regularity coefficients, on compact manifolds without boundary. We consider the eigenvalues $-\lambda^2$ and eigenfunctions $\phi$ for an equation

$$(1) \quad d^*(a \, d\phi) + \lambda^2 \rho \phi = 0. \tag{1}$$

Here we assume $\rho > 0$ is a real, positive measurable function, and $a_x : T^*_x(M) \to T_x(M)$ is the transformation associated to a real symmetric form on $T^*_x(M)$, also strictly positive and measurable in $x$. The manifold $M$ and volume form $dx$ are assumed smooth, and $d^*$ is the transpose of the differential $d$ with respect to $dx$. This setting includes the most general elliptic second order operator on $M$, assumed self-adjoint with respect to some measurable volume form $\rho \, dx$, and assumed to annihilate constants, and hence of the form $\rho^{-1}d^*ad$. For limited regularity $a$ and $\rho$ we pose the problem as above to avoid domain considerations.

If we consider the real quadratic forms

$$Q_0(f, g) = \int_M f \, g \, \rho \, dx, \quad Q_1(f, g) = Q_0(f, g) + \int_M a(df, dg) \, dx,$$

then

$$Q_0(f, f) = \|f\|^2_{L^2(M, \rho \, dx)}, \quad Q_1(f, f) \approx \|f\|_{H^1(M)}^2,$$

hence $Q_0$ is compact relative to $Q_1$ by Rellich’s lemma. By the standard argument of simultaneously diagonalizing $Q_0$ and $Q_1$, there exists a complete orthonormal basis $\phi_j$ for $L^2(M, \rho \, dx)$ consisting of eigenfunctions for (1), with $\lambda_j \to \infty$.

The object of this paper is to establish bounds on the $L^2 \to L^q$ operator norm of the unit-width spectral projectors for (1). Let $\Pi_{\lambda}$ be the projection of $L^2(M, \rho \, dx)$ onto the subspace spanned by the eigenfunctions of (1) for which $\lambda_j \in [\lambda, \lambda + 1]$. In

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the case that the coefficients $\rho$ and $a$ are $C^\infty$, the following estimates hold, and are best possible in terms of the exponent of $\lambda$,

\begin{align}
\|\Pi_\lambda f\|_{L^q(M)} &\leq C \lambda^{\frac{n-1}{2}(\frac{1}{q} - \frac{1}{2})} \|f\|_{L^2(M)}, & 2 \leq q \leq q_n, \\
\|\Pi_\lambda f\|_{L^q(M)} &\leq C \lambda^{n(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}} \|f\|_{L^2(M)}, & q_n \leq q \leq \infty,
\end{align}

where

$$q_n = \frac{2(n+1)}{n-1}$$

For $C^\infty$ metrics the estimates at $q = q_n$ are due to Sogge [8]. The estimate for $q = \infty$ is related to the spectral counting remainder estimates of Avakumović-Levitin-Hörmander; it can also be obtained from Sogge’s estimate by Sobolev embedding. The case $q = 2$ is of course trivial, and all other values of $q$ follow from these endpoints by interpolation.

In [5], both estimates (2) and (3) were established on the full range of $q$ for the case that both $a$ and $\rho$ are of class $C^1$. On the other hand, Smith-Sogge [6] and Smith-Tataru [7] constructed examples, for each $0 < s < 2$, of functions $a$ and $\rho$ with coefficients of class $C^s$ (Lipschitz in case $s = 1$) for which there exist eigenfunctions $\phi_\lambda$ such that for all $q \geq 2$

$$\|\phi_\lambda\|_{L^q(M)} \geq C \lambda^{\frac{n-1}{2}(\frac{1}{q} - \frac{1}{2})} \|\phi_\lambda\|_{L^2(M)},$$

where $C > 0$ is independent of $\lambda$, and where

$$\sigma = \frac{2-s}{2+s}$$

For $2 < q < \frac{2(n+2s-1)}{n-1}$, this shows that the spectral projection estimates for $C^s$ metrics with $s < 2$ can be strictly worse than in the $C^2$ case.

In this paper, we consider the case of coefficients $a$ and $\rho$ of class $C^s$ for $1 \leq s < 2$ (Lipschitz in case $s = 1$.) We start by establishing the following bound, which by the examples of [6] is best possible on the indicated range of $q$.

**Theorem 1.** Assume that the coefficients $a$ and $\rho$ are either of class $C^s$ for some $1 < s < 2$, or Lipschitz class if $s = 1$. Let $\Pi_\lambda$ denote the $L^2$-projection onto the subspace spanned by eigenfunctions of (1) with $\lambda_j \in [\lambda, \lambda + 1]$. Then

$$\|\Pi_\lambda f\|_{L^q(M)} \leq C \lambda^{\frac{n-1}{2}(\frac{1}{q} - \frac{1}{2})}(1+\sigma) \|\phi_\lambda\|_{L^2(M)},$$

where $C > 0$ is independent of $\lambda$, and where

$$\sigma = \frac{2-s}{2+s}$$

For $2 < q < \frac{2(n+2s-1)}{n-1}$, this shows that the spectral projection estimates for $C^s$ metrics with $s < 2$ can be strictly worse than in the $C^2$ case.

In this paper, we consider the case of coefficients $a$ and $\rho$ of class $C^s$ for $1 \leq s < 2$ (Lipschitz in case $s = 1$.) We start by establishing the following bound, which by the examples of [6] is best possible on the indicated range of $q$.

**Theorem 2.** Let $B_R \subset M$ be a ball of radius $R = \lambda^{-\sigma}$. Then under the same conditions as Theorem 1

\begin{align}
\|\Pi_\lambda f\|_{L^q(B_R)} &\leq C \lambda^{n(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}} \|f\|_{L^2(M)}, & q_n \leq q \leq \infty.
\end{align}
Interpolating with the trivial $L^2$ estimate establishes the estimate (2) on such balls $B_R$. Since the constant $C$ in (4) is uniform for all balls $B_R$, we obtain the same global $L^2 \to L^\infty$ mapping properties in the case of Lipschitz coefficients as in the case of smooth coefficients,

$$
\|\Pi_\lambda f\|_{L^\infty(M)} \leq C \lambda^{\frac{n}{q} - \frac{1}{2} + \frac{n}{2}} \|f\|_{L^2(M)} .
$$

A corollary of this result is the Hörmander multiplier theorem on compact manifolds for functions of elliptic operators with Lipschitz coefficients, as shown by results of Duong-Ouhabaz-Sikora [1], section 7.2. We note that, in related work, Ivrii [2] has obtained the sharp spectral counting remainder estimate for operators with coefficients of regularity slightly stronger than Lipschitz.

The proof of Theorem 2 that we will present requires that $q$ be not too large, but in all dimensions works for $q = q_n$. We therefore show here how heat kernel estimates permit us to deduce (4) for all $q \geq q_n$ from the case $q = q_n$. For this, let $H_\lambda$ denote the heat kernel at time $\lambda^{-2} \leq 1$ for the diffusion system associated to (1). By Theorem 6.3 of Saloff-Coste [4], the integral kernel $h_\lambda$ of $H_\lambda$ satisfies

$$
|h_\lambda(x, y)| \leq C \lambda^n \exp(-c \lambda^2 d(x, y)^2).
$$

By Young’s inequality, then for $q_n \leq q \leq \infty$

$$
\|\Pi_\lambda f\|_{L^q(B_R)} \leq C \lambda^n \left(1 + \frac{1}{q}\right) \|H_\lambda^{-1}\Pi_\lambda f\|_{L^q(B_R^*)} + C \lambda^{-N} \|H_\lambda^{-1}\Pi_\lambda f\|_{L^q(M\setminus B_R^*)} \leq C \lambda^n \left(1 + \frac{1}{q}\right) \|f\|_{L^2(M)}
$$

where we use (4) at $q = q_n$ with $B_R$ replaced by its double $B_R^*$, and the fact that $\|H_\lambda^{-1}\Pi_\lambda f\|_{L^2} \approx \|\Pi_\lambda f\|_{L^2}$ since $\exp(\lambda_j^2/\lambda^2) \approx 1$ for $\lambda_j \in [\lambda, \lambda + 1]$.

If we interpolate the estimate of Theorem 1 at $q = q_n$ with the estimate (5), then we obtain the following.

**Corollary 3.** Under the same conditions as Theorem 1

$$
\|\Pi_\lambda f\|_{L^q(M)} \leq C \lambda^{n\left(\frac{1}{q} - \frac{1}{2}\right) + \frac{n}{2}} \|f\|_{L^2(M)} , \quad q_n \leq q \leq \infty.
$$

For $q_n < q < \infty$, however, the exponent is strictly larger than that predicted by the examples of [6]. It is not currently known what the sharp exponent is for this range.

The key idea in our proof is that a $C^s$ function is well approximated on sets of diameter $R = \lambda^{-\sigma}$ by a $C^2$ function, up to an error which is suitably bounded when dealing with eigenfunctions localized to frequency $\lambda$. In effect, rescaling by $R$ reduces matters to a $C^2$ situation, where no-loss estimates hold. The loss of $\lambda^\frac{n}{2}$ comes from adding up the bounds over $R \approx R^{-1}$ disjoint sets.

This scaling parameter $R$ occurs in the examples of Smith-Sogge [6] and Smith-Tataru [7], The idea of scaling by $R$ to prove $L^p$ estimates was first used by Tataru in [9], to establish Strichartz-type estimates for time-dependent wave equations with $C^s$ coefficients, yielding improved existence theorems for a class of quasilinear hyperbolic equations.

**Notation.** By a $C^s$ function on $\mathbb{R}^n$, for $1 < s \leq 2$ we understand a continuously differentiable function $f$ such that

$$
\|f\|_{C^s} = \|f\|_{L^\infty(\mathbb{R}^n)} + \|Df\|_{L^\infty(\mathbb{R}^n)} + \sup_{h \in \mathbb{R}^n} |h|^{1-s} \|Df(\cdot + h) - Df(\cdot)\|_{L^\infty(\mathbb{R}^n)} < \infty.
$$
Thus, $C^s$ coincides with $C^{1,s-1}$ for $s \in \{1, 2\}$. For $s = 1$, we use $C^1$ to mean Lipschitz. For $0 < s < 1$ we take $C^s$ to be the standard Hölder class.

We use $d^*$ to denote the differential taking functions to covector fields, and $d^*$ its adjoint with respect to $dx$. When working on $\mathbb{R}^n$, $d = (\partial_1, \ldots, \partial_n)$, and $d^*$ is the standard divergence operator.

The notation $A \lesssim B$ means $A \leq C B$, where $C$ is a constant that depends only on the $C^s$ norm of $a$ and $\rho$, as well as on universally fixed quantities, such as the manifold $M$ and the non-degeneracy of $a$ and $\rho$. In particular, $C$ can be taken to depend continuously on $a$ and $\rho$ in the $C^s$ norm, so our estimates are uniform under small $C^s$ perturbations of $a$ and $\rho$.

### 2. Scaling Arguments

Our starting point is the following square-function estimate for solutions to the Cauchy problem. For $C^\infty$ coefficients this was established by Mockenhaupt-Seeger-Sogge [3]. The version we need for $C^{1,1}$ metrics is Theorem 1.3 of [5]. That theorem was stated under the condition $F = 0$ and for coefficients which are constant for large $x$, but these conditions are easily dropped by the Duhamel principle and a partition of unity argument.

**Theorem 4.** Suppose that $a$ and $\rho$ are defined globally on $\mathbb{R}^n$, and that

$$\|a^{|1/\gamma - \delta^0|}\|_{C^{1,1}(\mathbb{R}^n)} + \|\rho - 1\|_{C^{1,1}(\mathbb{R}^n)} \leq c_0,$$

where $c_0$ is a small constant depending only on $n$. Let $u$ solve the Cauchy problem

$$\rho(t(x)) \partial_t^0 u(t, x) = d^*(a(x) du(t, x)) = F(t, x), \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x).$$

Then

$$\|u\|_{L^q_t L^r_x([0, 1])} \lesssim \|u_0\|_{H^{1/2} + \|u_1\|_{H^{1/2} - 1}} + \|F\|_{L^1_t L^r_x}.$$

We first deduce the following corollary which is more useful for our purposes.

**Corollary 5.** Suppose that $f$ satisfies an equation on $\mathbb{R}^n$ of the form

$$\partial_t u + \mu^2 \rho_1 + 2 + g_2.$$

If $a$ and $\rho$ satisfy the condition of Theorem 4, then

$$\|d^*(a df) + \mu^2 \rho f - d^* g_1 + g_2 \|_{L^q, L^r} \lesssim \mu \|d^* f\|_{L^2} + \|f\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1} \|g_2\|_{L^2}.$$

**Proof.** Let $S_\epsilon = S_{\epsilon}(D)$ denote a smooth cutoff on the Fourier transform side to frequencies of size $|\xi| \leq r$. Let $a_\mu = S_{\epsilon \mu} a$, for $c$ to be chosen suitably small. Then

$$\|a - a_\mu\|_{L^2} \lesssim \epsilon^{-2} \mu^{-1} \|df\|_{L^2}, \quad \mu^{-1}(\mu - \mu_0)\|f\|_{L^2} \lesssim \epsilon^{-2} \mu \|f\|_{L^2},$$

and thus we may replace $a$ and $\rho$ by $a_\mu$ and $\rho_\mu$ at the expense of absorbing the above two terms into $g_1$ and $g_2$, which does not change the size of the right hand side of (7).

Next, let $f_{<\mu} = S_{\epsilon \mu} f$. Since

$$\|[S_{\epsilon \mu}, a_\mu]\|_{L^2 \to L^2} \lesssim (\epsilon \mu)^{-1},$$

and similarly for $[S_{\epsilon \mu}, \rho_\mu]$, we can absorb the commutator terms into $g_1$ and $g_2$, and since all terms are localized to frequencies less than $\mu$ we can write

$$d^*(a_\mu df_{<\mu}) + \mu^2 \rho_\mu f_{<\mu} = g_{<\mu},$$

(8)
where
\[ \|g_{<\mu}\|_{L^2} \lesssim \mu \|f\|_{L^2} + \|df\|_{L^2} + \mu \|g_1\|_{L^2} + \|g_2\|_{L^2} \]
Since \(\|d^*(a_\mu df_{<\mu})\|_{L^2} \lesssim (c\mu)^2\|f_{<\mu}\|_{L^2}\), for \(c\) suitably small the \(L^2\) norm of the left hand side of (8) is comparable to \(\mu^2\|f_{<\mu}\|_{L^2}\), hence we have
\[ \|f_{<\mu}\|_{L^2} \lesssim \mu^{-1} (\|f\|_{L^2} + \mu^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1}\|g_2\|_{L^2}) \]
Sobolev embedding now implies (7) if \(f\) is replaced on the left hand side by \(f_{<\mu}\). In fact there is a gain of \(\mu^{-\frac{1}{2}}\), since \(\frac{1}{q_0} = n\left(\frac{1}{2} - \frac{1}{q_0}\right) - \frac{1}{2}\).

If we let \(f_{>\mu} = f - S_{c_{\mu}} f\), then similar arguments let us write
\[ d^*(a_\mu df_{>\mu}) + \mu^2 \rho_\mu f_{>\mu} = d^*g_{>\mu} \]
where now \(g_{>\mu}\), like \(f_{>\mu}\), is frequency localized to frequencies larger than \(c^{-1} \mu\), and
\[ \|g_{>\mu}\|_{L^2} \lesssim \|f\|_{L^2} + \mu^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1}\|g_2\|_{L^2} \]
Taking the inner product of both sides of (9) against \(f_{>\mu}\) yields
\[ \|df_{>\mu}\|_{L^2}^2 - 4\mu^2\|f_{>\mu}\|^2_{L^2} \lesssim \|g_{>\mu}\|_{L^2}\|df_{>\mu}\|_{L^2} \]
and by the frequency localization of \(f_{>\mu}\) we obtain
\[ \|f_{>\mu}\|_{H^1} \lesssim \|f\|_{L^2} + \mu^{-1}\|df\|_{L^2} + \|g_1\|_{L^2} + \mu^{-1}\|g_2\|_{L^2} \]
Since \(n\left(\gamma - \frac{1}{q_0}\right) = \frac{1}{q_0} + \frac{1}{2} \leq 1\), Sobolev embedding yields (7) if \(f\) is replaced on the left hand side by \(f_{>\mu}\). As above, there is in fact a gain of \(\mu^{-\frac{1}{2}}\) for this term.

We now let \(f_\mu = S_{c_{\mu}} f - S_{c_{\mu}} f\), and as above write
\[ d^*(a_\mu df_\mu) + \mu^2 \rho_\mu f_\mu = g_\mu \]
where now \(f_\mu\) and \(g_\mu\) are localized to frequencies comparable to \(\mu\), and
\[ \|g_\mu\|_{L^2} \lesssim \mu \|f\|_{L^2} + \|df\|_{L^2} + \mu \|g_1\|_{L^2} + \|g_2\|_{L^2} \]
Setting \(u(t, x) = \cos(\mu t)f_\mu(x)\), we apply (6) to deduce
\[ \|f_\mu\|_{L^2} \lesssim \mu^{-\frac{1}{2}} \left( \|f_\mu\|_{L^2} + \mu^{-1}\|g_\mu\|_{L^2} \right) \]
which yields (7) for this term. \(\square\)

**Remark.** For future use, we note that in the proof of Corollary 5 the assumption that \(a \in C^{1,1}\) was used only at the last step, in order to deduce that (6) holds. The commutator and approximation bounds require only that \(a\) and \(\rho\) be Lipschitz. In particular, the bounds on \(f_{<\mu}\) and \(f_{>\mu}\) hold for Lipschitz \(a\) and \(\rho\).

**Corollary 6.** Let \(Q\) be a unit cube and \(Q^*\) its double. Suppose that \(a\) and \(\rho\) are bounded and measurable, and that there exist \(C^{1,1}\) functions \(\tilde{a}\) and \(\tilde{\rho}\) satisfying the conditions of Theorem 4 such that
\[ \|a - \tilde{a}\|_{L^\infty(Q^*)} + \|\rho - \tilde{\rho}\|_{L^\infty(Q^*)} \leq \mu^{-1} \]
Suppose that on \(Q^*\) we have
\[ d^*(a df) + \mu^2 \rho f = d^*g_1 + g_2 \]
Then
\[ \|f\|_{L^\infty(Q)} \lesssim \mu^{-\frac{1}{2}} \left( \|f\|_{L^2(Q^*)} + \mu^{-1}\|df\|_{L^2(Q^*)} + \|g_1\|_{L^2(Q^*)} + \mu^{-1}\|g_2\|_{L^2(Q^*)} \right) \]
The constant in the inequality is uniform for $\mu \geq 1$.

Proof. Let $\phi$ be a smooth function, equal to 1 on $Q$ and supported in $Q^\ast$. Then
\[ d^\ast(a \, d(\phi f)) + \mu^2 \rho(\phi f) = d^\ast [(a \, d\phi) f + \phi g_1] + [(a \, d\phi) \cdot df - (d\phi) \cdot g_1 + \phi g_2] \]
\[ = d^\ast \tilde{g}_1 + \tilde{g}_2 \]
where for $\mu \geq 1$
\[ \|\tilde{g}_1\|_{L^2} + \mu^{-1} \|\tilde{g}_2\|_{L^2} \lesssim \|f\|_{L^2(Q^\ast)} + \mu^{-1}\|df\|_{L^2(Q^\ast)} + \|g_1\|_{L^2(Q^\ast)} + \mu^{-1}\|g_2\|_{L^2(Q^\ast)} \]
One may similarly absorb $(a - \tilde{a})d(\phi f)$ into $\tilde{g}_1$, and $\mu^2(\rho - \tilde{\rho})(\phi f)$ into $\tilde{g}_2$. The result now follows from (7). \hfill \Box

Corollary 7. Suppose that $a$ and $\rho$ are of class $C^s$, with $0 \leq s \leq 2$, and that
\[ \|a^{ij} - \delta^{ij}\|_{C^s(\mathbb{R}^n)} + \|\rho - 1\|_{C^s(\mathbb{R}^n)} \leq c_0, \]
where $c_0$ is a small constant depending only on $n$. Suppose that $R = \lambda^{-\sigma}$, where $\sigma = \frac{2}{q_q^\ast - 1}$ and $\lambda \geq 1$. Assume $Q_R$ is a cube of sidelength $R$, $Q_R^\ast$ is its double, and on $Q_R^\ast$ the following equation holds
\[ d^\ast(a \, df) + \lambda^2 \rho f = d^\ast g_1 + g_2 \]
Then
\[ \|f\|_{L^{q_q^\ast}(Q_R^\ast)} \lesssim R^{-\frac{1}{2}} \lambda \frac{c_0}{\mu} \left( \|f\|_{L^2(Q_R^\ast)} + \lambda^{-1}\|df\|_{L^2(Q_R^\ast)} + R\|g_1\|_{L^2(Q_R^\ast)} + R\lambda^{-1}\|g_2\|_{L^2(Q_R^\ast)} \right). \]

Proof. We use the notation $f_R(x) = f(Rx)$. Then, for $\mu = R\lambda = \lambda^1^{-\sigma}$,
\[ d^\ast(a \, df_R) + \mu^2 \rho R \, f_R = R \, d^\ast g_1,R + R^2 \, g_2,R \]
holds on $Q^\ast$, with $Q$ a unit cube. If $\tilde{a} = S_{\mu^{1/2}}a_R$, then
\[ \|\tilde{a} - a_R\|_{L^\infty} \lesssim \mu^{-\frac{1}{q_q^\ast}} R^\ast\|a - I\|_{C^s} = c_0 \mu^{-1} \]
By the frequency localization, $\tilde{a}$ satisfies the conditions of Theorem 4. We may thus apply Corollary 6 to yield
\[ \|f_R\|_{L^{q_q^\ast}(Q)} \lesssim (R\lambda)^{\frac{1}{2}} \left( \|f_R\|_{L^2(Q^\ast)} + \lambda^{-1}\|(df)_R\|_{L^2(Q^\ast)} + R\|g_{1,R}\|_{L^2(Q^\ast)} + R\lambda^{-1}\|g_{2,R}\|_{L^2(Q^\ast)} \right) \]
Recalling that $\frac{1}{q_q^\ast} = n\left(\frac{1}{2} - \frac{1}{q_\ast}\right) - \frac{1}{2}$, this yields the corollary after rescaling. \hfill \Box

3. Proof of Theorem 1

The proof of Corollary 7 works for all $s \in [0,2]$, but the energy estimates of this section require that $a$ and $\rho$ be Lipschitz, hence we assume $s \geq 1$ for the remainder.

The projection $\Pi_\lambda f$ satisfies
\[ \|d^\ast(a \, d(\Pi_\lambda f)) + \lambda^2 \rho \Pi_\lambda f\|_{L^2(M,\rho dx)} \leq (2\lambda + 1) \|\Pi_\lambda f\|_{L^2(M,\rho dx)} \]
\[ \|d \Pi_\lambda f\|_{L^2(M,\rho dx)} \lesssim (\lambda + 1) \|\Pi_\lambda f\|_{L^2(M,\rho dx)} \]
hence Theorem 1 follows from showing that, if the following holds on $M$ (10)
\[ d^\ast(a \, df) + \lambda^2 \rho f = g \]
then uniformly for \( \lambda \geq 1 \)
\[
\|f\|_{L^\infty(M)} \lesssim \lambda^{1/4} \left( \|f\|_{L^2(M)} + \lambda^{-1} \|df\|_{L^2(M)} + \lambda^{-1} \|g\|_{L^2(M)} \right)
\]
Assume that (10) holds, and let \( \phi \) be a \( C^2 \) bump function on \( M \). Then
\[
d^*(a \, d(\phi f)) + \lambda^2 \rho \, f = f \, d^*(a \, d\phi) + (a \, d\phi, df) + \phi g
\]
Absorbing the terms on the right into \( g \) leaves the right hand side of (11) unchanged, hence by a partition of unity argument we may assume that \( f \) is supported in a suitably small coordinate neighborhood on \( M \).

We choose coordinate patches so that, in local coordinates, the conditions of Corollary 7 are satisfied after extending \( a \) and \( \rho \) to all of \( \mathbb{R}^n \). Thus, we have an equation of the form (10) on \( \mathbb{R}^n \), with \( f \) and \( g \) supported in a unit cube.

We next decompose \( f = f_{<\lambda} + f_{>\lambda} + f_{\Lambda} \) as in the proof of Corollary 5. As remarked following that proof, the bounds on \( f_{<\lambda} \) and \( f_{>\lambda} \) hold for \( a \) and \( \rho \) Lipschitz, hence we are reduced to considering \( f_{\Lambda} \), for which we have an equation
\[
d^*(a_{\Lambda} \, df_{\Lambda}) + \lambda^2 \rho_{\Lambda} \, f_{\Lambda} = g_{\Lambda}
\]
where \( a_{\Lambda} \) and \( \rho_{\Lambda} \) are localized to frequencies smaller than \( \lambda^2 \), and both \( f_{\Lambda} \) and \( g_{\Lambda} \) are localized to frequencies of size comparable to \( \lambda \).

We then decompose \( f_{\Lambda} = \sum_{j=1}^N \Gamma_j f_{\Lambda} \), where each \( \Gamma_j = \Gamma_j(D) \) is an order 0 multiplier, with symbol \( \Gamma_j(\xi) \) supported where \( |\xi| \approx \lambda \) and in a cone of suitably small angle. It then suffices to bound each \( \|\Gamma_j f_{\Lambda}\|_{L^\infty(Q)} \) by the right hand side of (11). Without loss of generality we consider a term with \( \Gamma(\xi) \) localized to a small cone about the \( \xi_1 \) axis.

We write
\[
d^*(a_{\Lambda} \, d\Gamma_j f_{\Lambda}) + \lambda^2 \rho_{\Lambda} \, \Gamma_j f_{\Lambda} = \Gamma_g + d^*[a_{\Lambda}, \Gamma] \, df_{\Lambda} + \lambda^2 \rho_{\Lambda}, \Gamma] \, f_{\Lambda}
\]
Simple commutator estimates show that the right hand side has \( L^2 \) norm bounded by \( \lambda \|f\|_{L^2} + \|g\|_{L^2} \), hence we are reduced to establishing
\[
\|f\|_{L^\infty(Q)} \lesssim \lambda^{1/4} \left( \|f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|df\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|g\|_{L^2(\mathbb{R}^n)} \right)
\]
for \( f \) satisfying the equation
\[
d^*(a_{\Lambda} \, df) + \lambda^2 \rho_{\Lambda} \, f = g
\]
where \( \widehat{f}(\xi) \) and \( \widehat{g}(\xi) \) are localized to \( |\xi| \approx \lambda \) and \( \xi \) in a small cone about the \( \xi_1 \) axis.

By Corollary 7, for any cube \( Q_R \) of sidelength \( R = \lambda^{-\sigma} \), we have
\[
\|f\|_{L^\infty(Q_R)} \lesssim \lambda^{1/4} \left( R^{-\frac{1}{2}} \|f\|_{L^2(Q_R)} + R^{-\frac{1}{2}} \lambda^{-1} \|df\|_{L^2(Q_R)} + R^\frac{1}{2} \lambda^{-1} \|g\|_{L^2(Q_R)} \right).
\]
Let \( S_R \) denote a slab of the form \( \{ x \in \mathbb{R}^n : |x_1 - c| \leq R \} \). By summing over cubes \( Q_R \) contained in \( S_R \), and noting \( R \leq 1 \), we obtain
\[
\|f\|_{L^\infty(S_R)} \lesssim \lambda^{1/4} \left( R^{-\frac{1}{2}} \|f\|_{L^2(S_R)} + R^{-\frac{1}{2}} \lambda^{-1} \|df\|_{L^2(S_R)} + \lambda^{-1} \|g\|_{L^2(S_R)} \right)
\]
We will show that
\[
R^{-\frac{1}{2}} (\|f\|_{L^2(S_R)} + \lambda^{-1} \|df\|_{L^2(S_R)}) \lesssim \|f\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|df\|_{L^2(\mathbb{R}^n)} + \lambda^{-1} \|g\|_{L^2(\mathbb{R}^n)}
\]
Given this, inequality (12) follows from (14) by adding over the \( R^{-1} = \lambda^\sigma \) disjoint slabs that intersect \( Q \). Also, the bound (13) implies the conclusion of Theorem 2 for \( q = g_n \) (hence for all \( q \) by the heat kernel arguments following that theorem.)
We establish (15) by energy inequality arguments. Let \( V \) denote the vector field
\[
V = 2(\partial_1 f) a_\lambda \, df + (\lambda^2 \rho_\lambda f^2 - \langle a_\lambda \, df, df \rangle) \vec{e}_1
\]
Then
\[
d^* V = 2(\partial_1 f) g + \lambda^2 (\partial_1 \rho_\lambda) f^2 - \langle (\partial_1 a_\lambda) \, df, df \rangle
\]
Applying the divergence theorem on the set \( x_1 \leq r \) yields
\[
\int_{x_1 = r} V_1 \, dx' \lesssim \lambda^2 \| f \|^2_{L^2(\mathbb{R}^n)} + \| df \|^2_{L^2(\mathbb{R}^n)} + \| g \|^2_{L^2(\mathbb{R}^n)}
\]
Since \( a_\lambda \) and \( \rho \) are pointwise close to the flat metric, we have pointwise that
\[
V_1 \geq \frac{3}{4} |\partial_1 f|^2 + \frac{3}{4} \lambda^2 |f|^2 - |\partial_x f|^2
\]
The frequency localization of \( \hat{f} \) to \( |\xi| \leq c \lambda \) yields
\[
\int_{x_1 = r} V_1 \, dx' \geq \frac{1}{2} \int_{x_1 = r} |df|^2 + \lambda^2 |f|^2 \, dx'
\]
Integrating this over \( r \) in an interval of size \( R \) yields (15). \( \square \)

References

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