STRICHARTZ ESTIMATES FOR THE WAVE EQUATION ON MANIFOLDS WITH BOUNDARY

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1. Introduction

Let (M, g) be a Riemannian manifold of dimension $n \geq 2$. Strichartz estimates are a family of space time integrability estimates on solutions $u(t,x) : (-T,T) \times M \to \mathbb{C}$ to the wave equation

$$\partial_t^2 u(t,x) - \Delta_{\mathrm{g}} u(t,x) = 0\,, \qquad u(0,x) = f(x)\,, \qquad \partial_t u(0,x) = g(x) \tag{1.1}$$

where Δ_{g} denotes the Laplace-Beltrami operator on (M, g). Local homogeneous Strichartz estimates state that

$$\|u\|_{L^p((-T,T);L^q(M))} \leq C \left(\|f\|_{H^\gamma(M)} + \|g\|_{H^{\gamma-1}(M)} \right) \tag{1.2}$$

where H^γ denotes the L^2 Sobolev space over M of order γ, and $2 \leq p \leq \infty$, $2 \leq q < \infty$ satisfy

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \qquad\qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \tag{1.3}$$

Estimates involving $q = \infty$ hold when $(n,p,q) \neq (3,2,\infty)$, but typically require the use of Besov spaces.

Strichartz estimates are well established on flat Euclidean space, where $M = \mathbb{R}^n$ and $\mathrm{g}_{ij} = \delta_{ij}$. In that case, one can obtain a global estimate with $T = \infty$; see for example Strichartz [27], Ginibre and Velo [9], Lindblad and Sogge [16], Keel and Tao [14], and references therein. However, for general manifolds phenomena such as trapped geodesics and finiteness of volume can preclude the development of global estimates, leading us to consider local in time estimates.

If M is a compact manifold without boundary, finite speed of propagation shows that it suffices to work in coordinate charts, and to establish local Strichartz estimates for variable coefficient wave operators on $\mathbb{R}^n$. Such inequalities were developed for operators with smooth coefficients by Kapitanski [13] and Mockenhaupt-Seeger-Sogge [18]. In this context one has the Lax parametrix construction, which yields the appropriate dispersive estimates. Strichartz estimates for operators with $C^{1,1}$ coefficients were shown by the second author in [21] and by Tataru in [29], the latter work establishing the full range of local estimates. Here the issue is more intricate as the lack of smoothness prevents the use of the Fourier integral operator machinery. Instead, wave packets or coherent state methods are used to construct parametrices for the wave operator.

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In this work, we consider the establishment of Strichartz estimates on a manifold with boundary, assuming that the solution satisfies either Dirichlet or Neumann homogeneous boundary conditions. Strichartz estimates for certain values of p, q were established by Burq-Lebeau-Planchon [5] using results from [25]; our work expands the range of indices p and q, and includes new estimates of particular interest for the critical nonlinear wave equation in dimensions 3 and 4. Our main result concerning Strichartz estimates is the following.

Theorem 1.1. *Let M be a compact Riemannian manifold with boundary. Suppose $2 < p \le \infty$, $2 \le q < \infty$ and (p, q, γ) is a triple satisfying*

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \qquad \begin{cases} \frac{3}{p} + \frac{n-1}{q} \le \frac{n-1}{2}, & n \le 4 \\ \frac{1}{p} + \frac{1}{q} \le \frac{1}{2}, & n \ge 4 \end{cases} \tag{1.4}$$

Then we have the following estimates for solutions u to (1.1) *satisfying either Dirichlet or Neumann homogeneous boundary conditions*

$$\|u\|_{L^p([-T,T];L^q(M))} \le C \left(\|f\|_{H^\gamma(M)} + \|g\|_{H^{\gamma-1}(M)} \right) \tag{1.5}$$

with C some constant depending on M and T.

A lemma of Christ-Kiselev [7] allows one to deduce inhomogeneous Strichartz estimates from the homogeneous estimates. In the following corollary, (r', s') are the Hölder dual exponents to (r, s), and the assumptions imply that a homogeneous $(H^{1-\gamma}, H^{-\gamma}) \to L^r L^s$ holds.

Corollary 1.2. *Let M be a compact Riemannian manifold with boundary. Suppose that the triples (p, q, γ) and $(r', s', 1-\gamma)$ satisfy the conditions of Theorem 1.1. Then we have the following estimates for solutions u to* (1.1) *satisfying either Dirichlet or Neumann homogeneous boundary conditions*

$$\|u\|_{L^p([-T,T];L^q(M))} \le C \left(\|f\|_{H^\gamma(M)} + \|g\|_{H^{\gamma-1}(M)} + \|F\|_{L^r([-T,T];L^s(M))} \right)$$

with C some constant depending on M and T.

For details on the proof of Corollary 1.2 using Theorem 1.1 and the Christ-Kiselev lemma we refer to Theorem 3.2 of [24], which applies equally well to Neumann conditions.

By finite speed of propagation, our results also apply to noncompact manifolds, provided that there is uniform control over the size of the metric and its derivatives in appropriate coordinate charts. In particular, we obtain local in time Strichartz estimates for the exterior in $\mathbb{R}^n$ of a compact set with smooth boundary, for metrics g which agree with the Euclidean metric outside a compact set. In this case one can obtain global in time Strichartz estimates under a nontrapping assumption. We refer to [24] for the case of odd dimensions, and Burq [4] and Metcalfe [17] for the case of even dimensions. See also [11].

For a manifold with strictly geodesically-concave boundary, the Melrose-Taylor parametrix yields the Strichartz estimates, for the larger range of exponents in (1.3) (not including endpoints) as was shown in [23]. If the concavity assumption is removed, however, the presence of multiply reflecting geodesics and their limits, gliding rays, prevent the construction of a similar parametrix.

Recently, Ivanovici [12] has shown that, when $n = 2$, (1.5) cannot hold for the full range of exponents in (1.3). Specifically, she showed that if $M \subset \mathbb{R}^2$ is a compact convex domain with smooth boundary then (1.5) cannot hold when $q > 4$ if $2/p + 1/q = 1/2$. It would be very interesting to determine the sharp range of exponents for (1.5) in any dimension $n \geq 2$.

The Strichartz estimates of Tataru [29] for Lipschitz metrics yield estimates in the boundary case, but with a strictly larger value of γ. The approach of [29] involves the construction of parametrices which apply over short time intervals whose size depends on frequency. Taking the sum over such sets generates a loss of derivatives in the inequality.

These ideas influenced the development of the spectral cluster estimates for manifolds with boundary appearing in [25]. Such estimates were established through squarefunction inequalities for the wave equation, which control the norm of $u(t, x)$ in the space $L^q(M; L^2(-T, T))$. These spectral cluster estimates were used in the work of Burq-Lebeau-Planchon [5] to establish Strichartz estimates for a certain range of triples (p, q, γ). The range of triples that can be obtained in this manner, however, is restricted by the allowed range of q for the squarefunction estimate. In dimension 3, for example, this restricts the indices to $p, q \geq 5$. In [5] similar estimates involving $W^{s,q}$ spaces were also established, and used in conjunction with the Strichartz estimates and boundary trace arguments to establish global well-posedness for the critical semilinear wave equation for $n = 3$. In the last two sections of this paper we shall present some new results concerning critical semilinear wave equations. Specifically, we shall obtain local well-posedness and global existence for small data when $n = 4$, as well as a natural scattering result for $n = 3$.

The approach of this paper instead adapts the proof of the squarefunction inequalities in [25]. We utilize the parametrix construction of that paper, and establish the appropriate time-dispersion bounds on the associated kernel. This allows us to obtain the Strichartz estimates for a wider range of triples, including, for example, the important $L^4((-T, T); L^{12}(M))$ estimate in dimension 3, and the $L^3((-T, T); L^6(M))$ estimate in dimension 4.

The key observation in [25] is that u satisfies better estimates if it is microlocalized away from directions tangent to ∂M than if it is microlocalized to directions nearly tangent to ∂M. This is due to the fact that one can construct parametrices over larger time intervals as one moves to directions further away from tangent to ∂M. More precisely, the parametrix for directions at angle $\approx \theta$ away from tangent to ∂M applies for a time interval of size θ, which would normally yield a θ-dependent loss in the estimate. However, this loss can be countered by the fact that such directions live in a small volume cone in frequency space. For *sub-critical* estimates, i.e. where strict inequality holds in the second condition in (1.3), this frequency localization leads to a gain for small θ. The restriction on p and q in Theorem 1.1 arises from requiring this gain to counteract the loss from adding over the θ^{-1} disjoint time intervals on which one has estimates. Hence, while the range of p and q in our theorem is not known to be optimal, the restrictions are naturally imposed by the local nature of the parametrix construction in [25].

Notation. The expression $X \lesssim Y$ means that $X \le CY$ for some C depending only on the manifold, metric, and possibly the triple (p, q, γ) under consideration. Also, we abbreviate $L^p(I; L^q(U))$ by $L^pL^q(I \times U)$.

2. Homogeneous Strichartz Estimates

The proof of Strichartz estimates is a direct adaptation of the proof of squarefunction estimates in [25]. The difference is that Strichartz estimates result from time decay of the wave kernel, whereas squarefunction estimates result from decay with respect to spatial separation. Consequently, in [25] the wave equation was conically localized in frequency so as to become hyperbolic with respect to a space variable labelled x_1, and the equation factored so as to make x_1 the evolution parameter.

In order to maintain the convention that x_1 is the evolution parameter, in this section we set $x_1 = t$, and will use $x' = (x_2, \ldots, x_{n+1})$ to denote spatial varibles in $\mathbb{R}^n$. Thus $x = (x_1, x')$ is a variable on $\mathbb{R}^{1+n}$.

We work in a geodesic-normal coordinate patch near ∂M in which $x_n \ge 0$ equals distance to the boundary (the estimates away from ∂M follow from [13] and [18]). The coefficients of the metric $\mathrm{g}_{ij}(x')$ are extended to $x_n < 0$ in an even manner, and the solution $u(x)$ is extended evenly in the case of Neumann boundary conditions, and oddly in x_n in case of Dirichlet conditions. The extended solution then solves the extended wave equation on the open set obtained by reflecting the coordinate patch in x_n.

Setting $a^{11}(x) = \sqrt{g(x')}$, we now work with an equation

$$\sum_{i,j=1}^{n+1} D_i a^{ij}(x') D_j u(x) = 0$$

on an open set symmetric in x_n. A linear change of coordinates, and shrinking the patch if necessary, reduces to considering coefficients $a^{ij}(x)$ which are pointwise close to the Minkowski metric on the unit ball in $\mathbb{R}^{1+n}$, and defined globally so as to equal that metric outside the unit ball.

Following [25, §2], the solution u is then localized in frequency to a conic set where $|\xi'| \approx |\xi_1|$. On the complement of this set the operator is elliptic, and the Strichartz estimates follow from elliptic regularity and Sobolev embedding. As in section 7 of [25], one uses the fact that the coefficients are smooth in all variables but x_n, and Sobolev embedding can be accomplished using at most one derivative in the x_n direction.

The next step is to take a Littlewood-Paley dyadic decomposition $u = \sum_{k=1}^{\infty} u_k$ with $\widehat{u}_k$ localized in frequency to shells $|\xi'| \approx 2^k$. One lets $a_k^{ij}(x)$ denote the coefficients frequency localized in the x' variables to $|\xi'| \le 2^k$, and factorizes

$$\sum_{i,j=1}^{n+1} a_k^{ij}(x)\xi_i\xi_j = a_k^{11}(x)\big(\xi_1 + p_k(x,\xi')\big)\big(\xi_1 - p_k(x,\xi')\big)\,,$$

where $p_k(x,\xi') \approx |\xi'|$. Just as in [25, §2], Theorem 1.1 is reduced to establishing, uniformly over $\lambda = 2^k$, bounds of the form

$$\|u_\lambda\|_{L^p_{x_1} L^q_{x'}(|x|\le 1)} \lesssim \lambda^\gamma \big(\|u_\lambda\|_{L^\infty L^2} + \|F_\lambda\|_{L^2}\big)\,, \quad D_1 u_\lambda - P_\lambda(x,D')u = F_\lambda\,. \tag{2.1}$$

Here, $P_\lambda(x,D') = \frac12 p_\lambda(x,D') + \frac12 p_\lambda(x,D')^*$, and the symbol $p_\lambda(x,\xi')$ can be taken frequency localized in x' frequencies to $|\xi'| \le \lambda$, and $p_\lambda(x,\xi') = |\xi'|$ if $|\xi'| \not\approx \lambda$.

The setup is now the same as in [25], and the reductions of §3-§6 of that paper, specifically their n-dimensional analogues of §7, apply directly. This starts with a decomposition $u_\lambda = \sum_j u_j$ corresponding to a dyadic decomposition of $\widehat{u}_\lambda(\xi)$ in the ξ_n variable to regions $\xi_n \in [2^{-j-2}\lambda, 2^{-j+1}\lambda]$ where $\lambda^{-1/3} \le 2^{-j} \le 1$.

If $2^{-j} \ge \frac18$, corresponding to non-tangential reflection, then the estimates will follow as the case for $2^{-j} = \frac18$, so we restrict attention to the case $2^{-j} \le \frac18$. Since $|\xi'| \approx \lambda$, this implies that some remaining variable is $\approx \lambda$, and after rotation we assume that $\widehat{u}_j(x_1,\xi')$ is supported in a set

$$\{\xi : \xi_{n+1} \approx \lambda,\ |\xi_j| \le c\lambda,\ j = 2,\dots,n-1,\ \textit{and}\ \xi_n \approx \theta_j\lambda\}$$

where $\lambda^{-1/3} \le \theta_j \le \frac18$.

The proof establishes good bounds on the term u_j over time intervals of length θ_j. Precisely, let $S_{j,k}$, $|k| \le \theta_j^{-1}$, denote the time slice $x_1 \in [k\epsilon\theta_j, (k+1)\epsilon\theta_j]$. In analogy with [25, Theorem 3.1], we establish the bound

$$\|u_j\|_{L^p_{x_1} L^q_{x'}(S_{j,k})} \lesssim \lambda^\gamma \theta_j^{\sigma(p,q)} c_{j,k} \tag{2.2}$$

where $c_{j,k}$ satisfies the nested summability condition [25, (3.1)], and where

$$\sigma(p,q) = \begin{cases} (n-1)(\frac12 - \frac1q) - \frac2p\,, & (n-2)(\frac12-\frac1q) \le \frac2p \\ \frac12 - \frac1q\,, & (n-2)(\frac12-\frac1q) \ge \frac2p \end{cases}$$

Adding over the θ_j^{-1} disjoint slabs intersecting $|x_1| \le 1$, the simple uniform bounds on $c_{j,k}$ yield

$$\|u_j\|_{L^p_{x_1} L^q_{x'}(|x_1|\le 1)} \lesssim \lambda^\gamma \theta_j^{\sigma(p,q)-1/p}\big(\,\|u_\lambda\|_{L^\infty L^2} + \|F_\lambda\|_{L^2}\big)\,.$$

The θ_j take on dyadic values less than 1, and provided $\sigma(p,q) > 1/p$, one can sum over j to obtain (2.1). In case $\sigma(p,q) = 1/p$ one can also sum the series, using the nested summability condition [25, (3.1)], together with the branching argument on [25, page 118], to yield (2.1). Note that the restrictions on (p,q) in Theorem 1.1 are precisely that $\sigma(p,q) \ge 1/p$.

Estimate (2.2) is established through the parametrix construction from [22], together with the use of the V^p_2 spaces of Koch-Tataru [15]. Precisely, one rescales $\mathbb{R}^{1+n}$ by θ_j, and considers the symbol

$$q(x,\xi') = \theta_j p_j(\theta_j x, \theta_j^{-1}\xi')\,,$$

where p_j is such that $\widehat{q}(x_1,\zeta,\xi')$ is supported in $|\zeta| \le c\mu^{1/2}$, where $\mu = \theta_j\lambda$ is the frequency scale at which $u_j(\theta_j x)$ is localized. Fix $u(x) = u_j(\theta_j x)$ and $\theta_j = \theta$, where now $1 \ge \theta \ge \mu^{-1/2}$. One writes

$$D_1 u - q(x,D')u = F + G$$

where G arises from the error term $(p - p_j)u_j$. The bound (2.2) is a consequence of the following bound (for a global $\varepsilon > 0$)

$$\|u\|_{L^p_{x_1} L^q_{x'}(|x_1|\le\varepsilon)} \lesssim \mu^\gamma \theta^{\sigma(p,q)} \Big(\|u\|_{L^\infty L^2(S)} + \|F\|_{L^1 L^2(S)} + \mu^{\frac14}\theta^{\frac12} \|\langle \mu^{\frac12} x_2\rangle^{-1} u\|_{L^2(S)} + \mu^{-\frac14}\theta_j^{-\frac12} \|\langle \mu^{\frac12} x_2\rangle^2 G\|_{L^2(S)} \Big), \tag{2.3}$$

and for $\theta = \mu^{-\frac12}$

$$\|u\|_{L^p_{x_1} L^q_{x'}(|x_1|\le\varepsilon)} \lesssim \mu^\gamma \theta^{\sigma(p,q)} \Big(\|u\|_{L^\infty L^2(S)} + \|F + G\|_{L^1 L^2(S)} \Big). \tag{2.4}$$

The solution u is written as a superposition of terms, each of which is product of $\chi_I(x_1)$, for an interval $I \subset [-\varepsilon, \varepsilon]$, with a functions whose wave-packet transform is invariant under the Hamiltonian flow of $q(x, \xi')$. The wave-packet transform, which acts in the x' variables, is a simple modification of the Gaussian transform used by Tataru [29] to establish Strichartz estimates for rough metrics; see also [30]. Precisely, set

$$\big(T_\mu f\big)(x', \xi') = \mu^{n/4} \int e^{-i\langle \xi', y'-x'\rangle} \, g\big(\mu^{\frac12}(y' - x')\big) \, f(y') \, dy' \, .$$

The base function g is taken to be of Schwartz class with $\widehat{g}$ supported in a ball of small radius. Thus, $\tilde u(x, \xi') = [T_\mu u(x_1, \cdot)](x', \xi')$ has the same localization in ξ' as does $\widehat{u}(x_1, \xi')$.

By Lemma 4.4 of [25] one can write

$$\Big(d_1 - d_{\xi'} q(x, \xi') \cdot d_{x'} + d_{x'} q(x, \xi') \cdot d_{\xi'}\Big) \tilde u(x, \xi') = \tilde F(x, \xi') + \tilde G(x, \xi') \, .$$

By variation of parameters and the use of V^p_2 spaces, one reduces matters to establishing estimates for solutions invariant under the flow. The use of the V^p_2 spaces from [15] requires $p > 2$, which is implied by the conditions of Theorem 1.1.

Let $\Theta_{t,s}$ denote the Hamiltonian flow of $q(x, \xi')$, from $x_1 = s$ to $x_1 = t$. Then the bounds (2.3)-(2.4) are consequences of the following, which is the analogue of Theorem 7.2 of [25].

Theorem 2.1. *Suppose that $f \in L^2(\mathbb{R}^{2n})$ is supported in a set of the form*

$$\{\xi : \xi_{n+1} \approx \mu \, , |\xi_j| \le c\mu \, , j = 2, \dots, n-1, \ \text{and} \ \xi_n \approx \theta\mu\}$$

or

$$\{\xi : \xi_{n+1} \approx \mu \, , |\xi_j| \le c\mu \, , j = 2, \dots, n-1, \ \text{and} \ |\xi_n| \le \mu^{\frac12}\}$$

in case $\theta = \mu^{-1/2}$.

If $Wf(x_1, x') = T^*_\mu\big[f \circ \Theta_{0,x_1}\big](x')$, then for admissible (p, q, γ)

$$\|Wf\|_{L^p_{x_1} L^q_{x'}(|t|\le\varepsilon)} \lesssim \mu^\gamma \theta^{\sigma(p,q)} \, \|f\|_{L^2(\mathbb{R}^{2n})} \, .$$

Proof. The function Wf is frequency localized to $\xi_n \approx \theta$ and $|\xi| \approx \mu\theta$ (respectively $|\xi_n| \le \mu^{-\frac12}$ when $\theta \approx \mu^{-\frac12}$). By duality, it suffices to show the estimate

$$\|WW^* F\|_{L^p L^q} \lesssim \mu^{2\gamma} \theta^{2\sigma(p,q)} \|F\|_{L^{p'} L^{q'}} . \tag{2.5}$$

for ξ'-frequency localized F. We use t and s in place of x_1 and y_1 for ease of notation. Then the operator WW^* applied to ξ'-localized F agrees with integration against the kernel

$$K(t,x';s,y') = \mu^{\frac{n}{2}}\int e^{i\langle\zeta,x'-z\rangle - i\langle\zeta_{s,t},y'-z_{s,t}\rangle} g(\mu^{\frac{1}{2}}(x'-z))\, g(\mu^{\frac{1}{2}}(y'-z_{s,t}))\,\beta_\theta(\zeta)\,dz\,d\zeta$$

where $(z_{s,t},\zeta_{s,t}) = \Theta_{s,t}(z,\zeta)$. To align with the notation that $x' = (x_2,\dots,x_{n+1})$ denote the space parameters, we take $\zeta = (\zeta_2,\dots,\zeta_{n+1})$. Then $\beta_\theta(\zeta)$ is a smooth cutoff to the set

$$\{\zeta : \zeta_{n+1}\approx\mu\,, |\zeta_j|\le c\mu\,, j=2,\dots,n-1, \text{ and } \zeta_n\approx\theta\mu\}$$

(respectively $|\zeta_n|\le\mu^{-\frac{1}{2}}$ in case $\theta=\mu^{-\frac{1}{2}}$.)

Analogous to [25, (7.1)-(7.2)], we establish the inequalities

$$\left\|\int K(t,x';s,y')f(y')\,dy'\right\|_{L^2_{x'}} \lesssim \|f\|_{L^2_{y'}}. \tag{2.6}$$

and

$$\left\|\int K(t,x';s,y')f(y')dy\right\|_{L^\infty_{x'}} \lesssim \mu^n\theta\,(1+\mu|t-s|)^{-\frac{n-2}{2}}(1+\mu\theta^2|t-s|)^{-\frac{1}{2}}\|f\|_{L^1_{y'}} \tag{2.7}$$

Interpolation then yields that

$$\left\|\int K(t,x';s,y')f(y')\,dy'\right\|_{L^q_{x'}}$$
$$\lesssim (\mu^n\theta)^{1-\frac{2}{q}}(1+\mu|t-s|)^{-\frac{n-2}{2}(1-\frac{2}{q})}(1+\mu\theta^2|t-s|)^{-\frac{1}{2}(1-\frac{2}{q})}\|f\|_{L^{q'}_{y'}}$$

In the case $\frac{n-2}{2}(1-\frac{2}{q})\le\frac{2}{p}\le\frac{n-1}{2}(1-\frac{2}{q})$, the exponent in the third factor on the right can be replaced by $\frac{n-2}{2}(1-\frac{2}{q})-\frac{2}{p}\le 0$, showing that

$$\left\|\int K(t,x';s,y')f(y')\,dy'\right\|_{L^q_{x'}} \lesssim \mu^{2\gamma}\theta^{2((n-1)(\frac{1}{2}-\frac{1}{q})-\frac{2}{p})}|t-s|^{-\frac{2}{p}}\|f\|_{L^{q'}_{y'}}$$

In the case $\frac{n-2}{2}(1-\frac{2}{q})\ge\frac{2}{p}$, we can ignore the last factor and obtain the bound

$$\left\|\int K(t,x';s,y')f(y')\,dy\right\|_{L^q_{x'}} \lesssim \mu^{2\gamma}\theta^{2(\frac{1}{2}-\frac{1}{q})}|t-s|^{-\frac{2}{p}}\|f\|_{L^{q'}_{y'}}$$

In both cases, the Hardy-Littlewood-Sobolev inequality then establishes (2.5).

The inequality (2.6) is estimate [25, (7.1)], which follows from the fact that T_μ is an isometry and $\Theta_{t,s}$ is a measure-preserving diffeomorphism. Hence it suffices to prove (2.7). As in [25], we consider two cases.

In the case $\mu\theta^2|t-s|\ge 1$, we fix $\overline{\theta}\le\theta$ so that $\mu\overline{\theta}^2|t-s|=1$, and decompose $\beta_\theta(\zeta)$ into a sum of cutoffs $\beta_j(\zeta)$, each of which is localized to a cone of angle $\overline{\theta}$ about some direction ζ_j. The proof of [25, Theorem 5.4] yields that

$$|K_j(t,x';s,y')| \lesssim \mu^n\overline{\theta}^{n-1}\big(1+\mu\overline{\theta}\,|y'-x'_{s,t,j}|\big)^{-N},$$

where $x'_{s,t,j}$ is the space component of $\Theta_{s,t}(x,\zeta_j)$. For each fixed (s,t) the $x'_{s,t,j}$ are a $(\mu\overline{\theta})^{-1}$ separated set, and adding over j yields the desired bounds, since in this case

$$\mu^n\overline{\theta}^{n-1} = \mu^{\frac{n+1}{2}}\,|t-s|^{-\frac{n-1}{2}} = \mu^n\theta\,(1+\mu|t-s|)^{-\frac{n-2}{2}}(1+\mu\theta^2|t-s|)^{-\frac{1}{2}}$$

In case $\mu\theta^2|t-s|\le 1$, we let $\overline{\theta}\ge\theta$ be given by

$$\overline{\theta} = \min\bigl(\,\mu^{-\frac12}|t-s|^{-\frac12}\,,\,1\,\bigr)\,.$$

Following the proof of [25, (7.2)], we set $\zeta''=(\zeta_2,\dots,\zeta_{n-1},\zeta_{n+1})$, and let β_j be a partition of unity in cones of angle $\overline{\theta}$ on $\mathbb{R}^{n-1}$. We then decompose

$$\beta_\theta(\zeta) = \sum_j \beta_\theta(\zeta)\,\beta_j(\zeta'')\,,$$

and let $K=\sum_j K_j$ denote the corresponding kernel decomposition.

The arguments on page 152 of [25] yield

$$|K_j(t,x';s,y')| \lesssim \mu^n\overline{\theta}^{n-2}\,\theta\,\bigl(\,1+\mu\overline{\theta}\,|(y'-x'_{s,t,j})_{2,\dots,n-1}|\,\bigr)^{-N}\,.$$

The $x'_{s,t,j}$ are $(\mu\overline{\theta})^{-1}$ separated in the $(2,\dots,n-1)$ variables as j varies, and summing over j yields

$$|K(t,x';s,y')| \lesssim \mu^n\theta\,\overline{\theta}^{n-2} \approx \mu^n\theta\,(1+\mu|t-s|)^{-\frac{n-2}{2}}(1+\mu\theta^2|t-s|)^{-\frac12}\,. \quad \square$$

3. Applications to semilinear wave equations

As an application, we consider the following family of semilinear wave equations with defocusing nonlinearity

$$\partial_t^2 u - \Delta u + |u|^{r-1}u = 0 \qquad (u,\partial_t u)|_{t=0}=(f,g) \qquad u|_{\partial M}=0, \tag{3.1}$$

or

$$\partial_t^2 u - \Delta u + |u|^{r-1}u = 0 \qquad (u,\partial_t u)|_{t=0}=(f,g) \qquad \partial_\nu u|_{\partial M}=0, \tag{3.2}$$

We will be mostly interested in the range of exponents $r<1+\frac{4}{n-2}$ (energy subcritical) and $r=1+\frac{4}{n-2}$ (energy critical).

In the boundaryless case where $\Omega=\mathbb{R}^n$, the first results for the critical wave equation were obtained by Grillakis [10]. He showed that when $n=3$ there are global smooth solutions of the critical wave equation, $r=5$, if the data is smooth. Shatah and Struwe [20] extended his theorem by showing that there are global solutions for data lying in the energy space $H^1\times L^2$. They also obtained results for critical wave equations in higher dimensions.

For the case of obstacles, the first results are due to Smith and Sogge [23]. They showed that Grillakis' theorem extends to the case where Ω is the compliment of a smooth compact obstacle and Dirichlet boundary conditions are imposed, i.e. (3.1) for $r=5$. Recently this result was extended to the case of arbitrary domains in $\Omega\subset\mathbb{R}^3$ and data in the energy space by Burq, Lebeau and Planchon [5]. The case of nonlinear critical Neumann-wave equations in 3-dimensions, (3.2), was subsequently handled by Burq and Planchon [6].

The proofs of the results for arbitrary domains in 3-dimensions used two new ingredients. First, the estimates of Smith and Sogge [25] for spectral clusters turned out to be strong enough to prove certain Strichartz estimates for the linear wave equations with either Dirichlet or Neumann boundary conditions. Specifially, Burq, Lebeau and Planchon [5] showed that one can control the $L^5W_0^{\frac{3}{10},5}$ norm of the solution of (1.1) over $[0,1]\times\Omega$ in terms of the energy norm of the data, assuming that Ω is compact. The other novelty was new estimates for the restriction of u to the boundary, specifically Proposition 3.2 in [5] and Proposition 3.1 in [6]. In the earlier case of convex obstacles and Dirichlet boundary conditions treated in [23] such estimates were not necessary since for the flux arguments that were used to treat the nonlinear wave equation (3.1), the boundary terms had a favorable sign. We remark that by using the results in Theorem 1.1, we can simplify the arguments in [5] and [6] since we now have control of the $L^4_tL^{12}_x([0,1]\times\Omega)$ norms of the solution of (1.1) in terms of the energy norm of the data. If this is combined with the aforementioned boundary estimates in [5] and [6] one can prove the global existence results in these papers by using the now-standard arguments that are found in [23] for convex obstacles, and [20] and [26] for the case where $\Omega=\mathbb{R}^3$. In the next section we shall show how these $L^4_tL^{12}_x$ and the weaker $L^5_tL^{10}_x$ estimates can be used to show that there is scattering for (3.1) when $n=3$, $r=5$ and Ω is the compliment of a star-shaped obstacle.

Let us conclude this section by presenting another new result. We shall show that the Strichartz estimates in Theorem 1.1 are strong enough to prove the following:

Theorem 3.1. *Suppose that $\Omega\subset\mathbb{R}^4$ is a domain with smooth compact boundary. If $1<r<3$ and $(f,g)\in(\dot H^1(\Omega)\cap L^{r+1}(\Omega))\times L^2(\Omega)$ then* (3.1) *and* (3.2) *have a unique global solution satisfying*

$$u\in C^0\big([0,T];\dot H^1(\Omega)\cap L^{r+1}(\Omega)\big)\cap C^1\big([0,T];L^2(\Omega)\big)\cap L^3_tL^6_x\big([0,T]\times\Omega\big)$$

for every $T>0$. If $r=3$ then the same result holds provided that the $(\dot H^1\cap L^4)\times L^2$ norm of (f,g) is sufficiently small.

The local existence results follow from the fact that Theorem 1.1 implies that if $(\partial_t^2-\Delta)v=F$ and v has either Dirichlet or Neumann boundary conditions then for $0<T<1$ there is a constant C so that

$$\|v\|_{L^3_tL^6_x((0,T)\times\Omega)}\le C\Big(\|v(0,\,\cdot\,)\|_{H^1}+\|\partial_tv(0,\,\cdot\,)\|_{L^2}+\int_0^T\|F(s,\,\cdot\,)\|_2\,ds\Big). \tag{3.3}$$

If Ω is the complement of a bounded set, then estimate (3.3) holds with H^1 replaced by $\dot H^1$, as can be seen by combining the estimates for the case of compact Ω with the global Strichartz estimates on $\mathbb{R}^4$, and using finite propagation velocity. Using this estimate the theorem follows from a standard convergent iteration argument with u in the space

$$X=C^0((0,T);\dot H^1(\Omega)\cap L^{r+1}(\Omega))\cap C^1((0,T);L^2(\Omega))\cap L^3_tL^6_x((0,T)\times\Omega),$$

and T being sufficiently small depending on the $(\dot H^1\cap L^{r+1})\times L^2$ norm of the initial data (f,g) of either (3.1) or (3.2) for $1<r<3$, and T depending on the data in the critical case $r=3$. For data of sufficiently small norm, one can obtain existence for $T=1$ for the critical case $r=3$. Together with energy conservation, the above yields global existence for $1<r<3$, and global existence for small data for $r=3$.

The analog of (3.3) when $n = 3$ involves $L^5_t L^{10}_x$ in the left. As we mentioned before, a stronger inequality involving $L^4_t L^{12}_x$ is valid when $n = 3$ by Theorem 1.1. Any such corresponding improvement of (3.3) when $n = 4$ would lead to a global existence theorem for arbitrary data for the critical case where $r = 3$, but, at present, we are unable to obtain such a result.

4. Scattering for star-shaped obstacles in 3-dimensions

We now consider solutions to the energy critical nonlinear wave equation in 3+1 dimensions in a domain $\Omega = \mathbb{R}^3 \setminus \mathcal{K}$ exterior to a compact, non-trapping obstacle $\mathcal{K}$ with smooth boundary

$$\begin{aligned} \Box u(t,x) = (\partial_t^2 - \Delta)u(t,x) &= -u^5(t,x), \qquad (t,x) \in \mathbb{R}\times\Omega \\ u\big|_{\mathbb{R}\times\partial\Omega} &= 0 \\ (\nabla u(t,\cdot), \partial_t u(t,\cdot)) &\in L^2(\Omega) \qquad t \in \mathbb{R} \end{aligned} \tag{4.1}$$

We restrict attention to real-valued solutions $u(t,x)$.

When $\mathcal{K}$ is a nontrapping obstacle, the estimates above, combined with those of Smith and Sogge [24] (see also Burq [4], Metcalfe [17]) imply the following estimate on functions $w(t,x)$ satisfying homogeneous Dirichlet boundary conditions

$$\begin{aligned} &\|w\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|w\|_{L^4(\mathbb{R};L^{12}(\Omega))} \\ &\qquad \le C\left(\|\,(\nabla_x w(0,\cdot), \partial_t w(0,\cdot))\,\|_{L^2(\Omega)} + \|\Box w\|_{L^1(\mathbb{R};L^2(\Omega))}\right). \end{aligned} \tag{4.2}$$

In this section, we show how these global estimates can be used to show that solutions to the nonlinear equation (4.1) above scatter to a solution to the homogeneous equation

$$\begin{aligned} \Box v(t,x) &= 0, \qquad (t,x) \in \mathbb{R}\times\Omega \\ v\big|_{\mathbb{R}\times\partial\Omega} &= 0 \\ (\nabla v(t,\cdot), \partial_t v(t,\cdot)) &\in L^2(\Omega) \qquad t \in \mathbb{R}. \end{aligned} \tag{4.3}$$

Let $\nu = \nu(x)$ denote the outward pointing unit normal vector to the boundary at $x \in \partial\mathcal{K}$. We call the obstacle $\mathcal{K}$ *star-shaped with respect to the origin* if $\nu(x)\cdot x \ge 0$ for all $x \in \partial\mathcal{K}$. Define the energy functional

$$E_0(v;t) = \frac{1}{2}\int_\Omega |\nabla_x v(t,x)|^2 + |\partial_t v(t,x)|^2 \, dx,$$

and recall that $t \mapsto E_0(v;t)$ is conserved whenever v is a solution to the homogeneous equation (4.3). We show the following:

Proposition 4.1. *Suppose u solves the nonlinear problem* (4.1) *and that $\mathcal{K}$ is star-shaped with respect to the origin. Then there exists unique solutions $v_\pm$ to* (4.3) *such that*

$$\lim_{t\to\pm\infty} E_0(u - v_\pm; t) = 0. \tag{4.4}$$

Moreover, u satisfies the space-time integrability bound

$$\|u\|_{L^5(\mathbb{R};L^{10}(\Omega))} + \|u\|_{L^4(\mathbb{R};L^{12}(\Omega))} < \infty. \tag{4.5}$$

When $\mathcal{K} = \varnothing$, this follows from the observations of Bahouri and Gérard [1]. We also remark that when $\mathcal{K}$ is convex, similar results for compactly supported, subcritical nonlinearities were obtained by Bchatnia and Daoulatli [3].

Attention will be restricted to the v_+ function, as symmetric arguments will yield the existence of a v_- asymptotic to u at $-\infty$. As observed in [1], we actually have that (4.4) follows as a consequence of (4.5). We first establish the existence of the wave operator, namely that for any solution v to (4.3), there exists a unique solution u to (4.1) such that

$$\lim_{t\to\infty} E_0(u - v; t) = 0.$$

Given (4.2), for any $\delta > 0$ we may select T large so that $\|v\|_{L^5([T,\infty);L^{10}(\Omega))} \leq \delta$. Given any $w(t, x)$ satisfying $\|w\|_{L^5([T,\infty);L^{10}(\Omega))} \leq \delta$, we have a unique solution to the linear problem

$$\Box\tilde{w} = -(v + w)^5$$
$$\lim_{t\to\infty} E_0(\tilde{w}; t) = 0$$

as the right hand side is in $L^1([T, \infty); L^2(\Omega))$. The estimate (4.2) then also ensures that

$$\|\tilde{w}\|_{L^5([T,\infty);L^{10}(\Omega))} \leq C\|(v + w)^5\|_{L^5([T,\infty);L^{10}(\Omega))} \leq 32C\delta^5.$$

Hence for δ sufficiently small, the map $w \mapsto \tilde{w}$ is a contraction in the ball of radius δ in $L^5([T, \infty); L^{10}(\Omega))$. The unique fixed point w can be uniquely extended over all of $\mathbb{R} \times \Omega$. Hence taking $u = v + w$ shows existence of the wave operator.

To see that the wave operator is surjective, we need a decay estimate which establishes that the nonlinear effects of the solution map for (4.1) diminish as time evolves.

Lemma 4.2. *Let $\mathcal{K}$ be star-shaped with respect to the origin. If $u(t, x)$ solves* (4.1)*, then the following decay estimate holds*

$$\lim_{t\to\infty} \frac{1}{6}\int_\Omega |u(t, x)|^6 \, dx = 0.$$

When $\mathcal{K} = \varnothing$, this is due to Bahouri and Shatah [2]. The proof below is essentially theirs, with slight modifications made to handle the boundary conditions. However, for the sake of completeness, we replicate the full proof below. We remark that the approach employs a nonlinear version of the vector field used by Morawetz [19] to prove decay of local energy for solutions to the linear wave equation exterior to a star-shaped obstacle.

To see that this implies the proposition, observe that given any $\varepsilon > 0$, there exists T sufficiently large such that

$$\sup_{t\geq T} \|u(t, \cdot)\|_{L^6} < \varepsilon.$$

Hence for any $S > T$ we obtain the following for any solution u to (4.1)

$$\begin{aligned}\|u\|_{L^5([T,S];L^{10}(\Omega))} + \|u\|_{L^4([T,S];L^{12}(\Omega))} &\le C\left(E + \|u^5\|_{L^1([T,S];L^2(\Omega))}\right)\\ &\le CE + C\varepsilon\|u\|_{L^4([T,S];L^{12}(\Omega))}\end{aligned}$$

where E denotes the conserved quantity

$$E = E(t) = \int_\Omega \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{6}|u(t,x)|^6\, dx.$$

A continuity argument now yields $\|u\|_{L^5([T,\infty);L^{10}(\Omega))} + \|u\|_{L^4([T,\infty);L^{12}(\Omega))} < 2CE$ and by a time reflection argument, (4.5) follows. However, this implies that the linear problem

$$\Box w = -u^5 \qquad\qquad \lim_{t\to\infty} E_0(w;t) = 0$$

admits a solution, showing that the wave operator is indeed surjective as $v = u - w$ is the desired solution to (4.3).

Proof of Lemma 4.2. By a limiting argument it suffices to consider smooth, classical solutions u which decay at infinity. We must show that for for any $\varepsilon_0 > 0$, there exists T_0 such that whenever $t \ge T_0$,

$$\frac{1}{6}\int_\Omega |u(t,x)|\, dx \le \varepsilon_0.$$

Consider the stress energy tensor associated with u (see Tao [28])

$$\begin{aligned}T^{00} &= \frac{1}{2}(\partial_t u)^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{6}u^6\\ T^{0j} &= -\partial_t u \partial_{x_j} u && 1 \le j \le 3\\ T^{jk} &= \partial_{x_j} u \partial_{x_k} u - \frac{\delta_{jk}}{2}(|\nabla u|^2 - (\partial_t u)^2 + \frac{1}{3}u^6) && 1 \le j,k \le 3.\end{aligned}$$

It can be checked that the divergence free property holds

$$\partial_t T^{00} + \partial_{x_j} T^{0j} = 0 \qquad \partial_t T^{0j} + \partial_{x_k} T^{jk} = 0$$

with the summation convention in effect. Taking the first of these identities and applying the divergence theorem to a region $\{0 \le t \le T, |x| \ge R + t\}$ (with $R > 0$ large enough so that $\mathcal{K} \subset B_R(0)$) we have

$$\begin{aligned}(4.6)\quad \int_{|x|\ge R+T} \frac{1}{2}|\partial_t u(T,x)|^2 + \frac{1}{2}|\nabla u(T,x)|^2 + \frac{1}{6}|u(T,x)|^6\, dx + \frac{1}{\sqrt{2}}\,\mathrm{flux}(0,T)&\\ \le \int_{|x|\ge R} \frac{1}{2}|\partial_t u(0,x)|^2 + \frac{1}{2}|\nabla u(0,x)|^2 + \frac{1}{6}|u(0,x)|^6\, dx&\end{aligned}$$

where

$$\mathrm{flux}(a,b) := \int_{M_b^a} \frac{1}{2}\left|\frac{x}{|x|}\partial_t u + \nabla u\right|^2 + \frac{|u|^6}{6}\, d\sigma$$

$$M_b^a := \{a < t < b, |x| = R + t\}$$

Since the solution has finite energy, we may select R large so that the right hand side of (4.6) is less than $\frac{\varepsilon_0}{40}$ (and again $\mathcal{K} \subset B_R(0)$). By time translation, $t \mapsto t + R$, it will suffice to show the existence of T_0 such that whenever $t > T_0$ we have

$$\frac{1}{6}\int_{x\in\Omega:|x|\le t} |u(t,x)|\, dx \le \frac{\varepsilon_0}{2}$$

(the additional smallness in the right hand side of (4.6) will be used later in the proof).

We now define the following vector field $X = (X^0, X^1, X^2, X^3)$ by contracting the stress-energy tensor with the null vector field $t\partial_t - x\cdot\nabla_x$ and adding a correction term

$$X^0 = tT^{00} - x_k T^{0k} + u\partial_t u$$
$$X^j = tT^{j0} - x_k T^{jk} - u\partial_{x_j} u \qquad 1 \le j \le 3.$$

The space-time divergence of X satisfies

$$\operatorname{div}(X) = -\frac{1}{3}u^6$$

We now apply the divergence theorem over the truncated cone $K_{T_1}^{T_2} = \{x \in \Omega : |x| \le t, T_1 \le t \le T_2\}$

$$\begin{aligned} 0 &= \int_{D(T_2)} X^0\,dx - \int_{D(T_1)} X^0\,dx - \int_{M_{T_1}^{T_2}} \left(X^0 - \sum_{j=1}^{3} \frac{x_j}{|x|} X^j \right) d\sigma \\ &+ \int_{K_{T_1}^{T_2}} \frac{|u|^6}{3}\,dx\,dt - \int_{\partial\Omega} \nu\cdot\langle X^1, X^2, X^3\rangle\,d\sigma \\ &= I + II + III + IV + V \end{aligned}$$

where $d\sigma$ denotes Lebesgue measure on the corresponding surface and $D(T_i) = \{x \in \Omega : |x| \le T_i\}$. The star-shaped assumption is crucial in controlling the last term V. Indeed, consider the restriction of the integrand in V to the $\partial\Omega(= \partial\mathcal{K})$ and observe that the Dirichlet boundary condition gives

$$\begin{aligned} \nu\cdot\langle X^1, X^2, X^3\rangle &= -\sum_{1\le j,k\le 3} \nu_j x_k \left(\partial_{x_j} u \partial_{x_k} u - \frac{\delta_{jk}}{2}|\nabla u|^2 \right) \\ &= -\left(\nu\cdot\nabla u\right)\left(x\cdot\nabla u\right) + \frac{1}{2}(\nu\cdot x)|\nabla u|^2. \end{aligned}$$

We have that ∇u is normal to $\partial\Omega$ and hence $|\nabla u|^2 = (\nu\cdot\nabla u)^2$. Treating x as a vector, we can project it on to the subspace orthogonal to ν obtaining

$$0 = \nabla u\cdot(x - (\nu\cdot x)\nu) = x\cdot\nabla u - (\nu\cdot x)(\nu\cdot\nabla u).$$

This now gives

$$\nu\cdot\langle X^1, X^2, X^3\rangle = -\frac{1}{2}\left(\nu\cdot x\right)\left(\nu\cdot\nabla u\right)^2 \le 0$$

and since $IV \ge 0$ is clear,

$$0 \ge I + II + III.$$

We now impose polar coordinates $(r, \omega) \in \mathbb{R}\times\mathbb{S}^2$ on the third term, writing

$$III = -\frac{1}{\sqrt{2}} \int_{M_{T_1}^{T_2}} \left(r(\partial_t u + \partial_r u)^2 + u(\partial_t u + \partial_r u) \right) d\sigma$$

where $\partial_r = \frac{x}{|x|}\cdot\nabla$ denotes the radial derivative. Next parameterize $M_{T_1}^{T_2}$ by $(r,\omega) \to (r, r\omega)$ and set $v(y) = u(|y|, y)$ (or $v(r\omega) = u(r, r\omega)$ in polar coordinates) so that we may write compactly

$$\begin{aligned} III &= -\int_{\mathbb{S}^2}\int_{T_1}^{T_2} r\left(\partial_r v + \frac{v}{r}\right)^2 r^2\, dr\, d\omega + \int_{\mathbb{S}^2}\int_{T_1}^{T_2} \frac{1}{2}\partial_r\left(r^2v^2\right)\, dr\, d\omega \\ &= -\int_{\mathbb{S}^2}\int_{T_1}^{T_2} r\left(\partial_r v + \frac{v}{r}\right)^2 r^2\, dr\, d\omega + \frac{1}{2}\int_{\mathbb{S}^2} T_2^2 v^2(T_2\omega)\, d\omega - \frac{1}{2}\int_{\mathbb{S}^2} T_1^2 v^2(T_1\omega)\, d\omega \end{aligned}$$

To handle the first term I, first observe that in polar coordinates

$$|\nabla u|^2 = (\partial_r u)^2 + \frac{1}{r^2}|\nabla_\omega u|^2 = (\partial_r u + \frac{1}{r}u)^2 + \frac{1}{r^2}|\nabla_\omega u|^2 - \frac{1}{r^2}\partial_r(ru^2).$$

Since $\mathcal{K}$ is star-shaped we may parameterize $\partial\Omega$ by $(r,\omega) = (\Psi(\omega),\omega)$ where Φ is a real valued function on $\mathbb{S}^2$. This allows us to write

$$\begin{aligned} I = \int_{D(T_2)} \frac{T_2}{2}&\left((\partial_t u)^2 + \left(\partial_r u + \frac{1}{r}u\right)^2 + \frac{1}{r^2}|\nabla_\omega u|^2 + \frac{1}{3}u^6\right) + r\left(\partial_r + \frac{1}{r}u\right)\partial_t u\, dx \\ &- \frac{1}{2}\int_{\mathbb{S}^2}\int_{\Psi(\omega)}^{T_2} T_2\partial_r(ru^2)\, dr\, d\omega \end{aligned} \tag{4.7}$$

Integrating by parts in the last term yields cancellation with one of the terms in III as the boundary condition gives $-\frac{1}{2}\int_{\mathbb{S}^2}\int_{\Psi(\omega)}^{T_2} T_2\partial_r(ru^2)\, dr\, d\omega = -\frac{1}{2}\int_{\mathbb{S}^2} T_2^2 v^2(T_2\omega)\, d\omega$. Similarly,

$$\begin{aligned} II = -\int_{D(T_1)} \frac{T_1}{2}&\left((\partial_t u)^2 + \left(\partial_r u + \frac{1}{r}u\right)^2 + \frac{1}{r^2}|\nabla_\omega u|^2 + \frac{1}{3}u^6\right) + r\left(\partial_r + \frac{1}{r}u\right)\partial_t u\, dx \\ &+ \frac{1}{2}\int_{\mathbb{S}^2} T_1^2 v^2(T_1\omega)\, d\omega \end{aligned}$$

In order to control remaining term in I we need to observe the following Hardy inequality, which holds in the exterior domain

$$\int_\Omega \frac{|u|^2}{|x|^2}\, dx \le 4\int_\Omega |\nabla u|^2\, dx. \tag{4.8}$$

To see this, we assume u is real-valued and denote the integral on left hand side as J and convert to polar coordinates

$$J = \int_{\mathbb{S}^2}\int_{\Psi(\omega)}^\infty (u(r\omega))^2\, dr\, d\omega = \int_{\mathbb{S}^2} ru(r\omega)^2\Big|_{\Psi(\omega)}^\infty\, d\omega - \int_{\mathbb{S}^2}\int_{\Psi(\omega)}^\infty 2u(\partial_r u)r\, dr\, d\omega.$$

The first term on the right is nonpositive (provided u exhibits sufficient decay at infinity) and Cauchy-Schwartz on the second term gives

$$J \le 2\sqrt{J}\left(\int_{\mathbb{S}^2}\int_{\Psi(\omega)}^\infty |\partial_r u|^2 r^2\, dr\, d\omega\right)^{\frac{1}{2}}.$$

The inequality (4.8) now follows.

We now observe that the first integral in (4.7) is bounded below by $T_2 \int_{D(T_2)} \frac{|u|^6}{6}\, dx$. Setting $T_2 = T > 0$ and $T_1 = \varepsilon T$ $(0 < \varepsilon < 1)$ and using the Hardy inequality (4.8) to control the first integral in II now yields

$$T \int_{D(T)} \frac{|u|^6}{6}\, dx \le C\varepsilon T E + \int_{\varepsilon T}^{T} \int_{\mathbb{S}^2} T \left(\partial_r v + \frac{v}{r}\right)^2 r^2\, d\omega\, dr.$$

Here E is the conserved quantity $E = E(t) = \int_\Omega T^{00}(t,x)\, dx$. We can now divide both sides of this inequality by T and choose ε sufficiently small so that $C\varepsilon E \le \varepsilon_0/4$, leaving us to control the integral involving v. However, by the proof of the Hardy inequality above we have

$$\int_{\varepsilon T}^{T} \int_{\mathbb{S}^2} \left(\partial_r v + \frac{v}{r}\right)^2 r^2\, d\omega\, dr \le 10 \int_{\varepsilon T}^{\infty} \int_{\mathbb{S}^2} (\partial_r v)^2\, r^2\, d\omega\, dr \le 10\ \mathrm{flux}(\varepsilon T, \infty) < \frac{\varepsilon}{4},$$

provided T is large enough so that $\varepsilon T > R$. $\square$

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