L^q BOUNDS FOR SPECTRAL CLUSTERS

HART F. SMITH

Abstract. In these notes, we review recent results concerning the L^p norm bounds for spectral clusters on compact manifolds. The type of estimates we consider were first established by Sogge [15] in the case of smooth metrics. Recent results of ours in [10] establish the same estimates under the assumption that the metric is $C^{1,1}$. It is known by examples of Smith-Sogge [12] that such estimates fail for $C^{1,\alpha}$ metrics if $\alpha < 1$, and we discuss methods for obtaining slightly weaker results in this setting also.

1. Introduction

The purpose of these notes is to give an overview of recent progress made towards understanding the behavior of solutions to scalar wave equations in the setting of metrics of limited differentiability. The calculus of Fourier integral operators and the associated asymptotic construction of the wave kernel do not apply in this situation. In the past decade, however, the introduction of wave-packet methods has permitted the construction of approximate solution operators which are sufficient to establish L^p norm estimates on solutions. These estimate and their proofs are the focus of this review.

The Strichartz estimates are perhaps the best known of the L^p estimates. In this paper, however, we pay more attention to estimates which yield L^p norm bounds on eigenfunctions and clusters of eigenfunctions. It is natural in this setting to stick to metrics that are time-independent, and we do so here. Time-dependent metrics are, of course, of great interest for Strichartz estimates and applications to quasilinear wave equations, and the results we discuss here all have analogues in the time-dependent setting.

We consider a compact manifold M without boundary, of dimension $n \geq 2$. Suppose given a second order operator P on M, which in local coordinates takes the form

$$\big(Pf\big)(x) = \rho(x)^{-1} \sum_{i,j=1}^{n} \partial_i \Big(\rho(x)\, \mathrm{g}^{ij}(x)\, \partial_j f(x)\Big)\,. \tag{1.1}$$

We assume that the function $\rho(x)$ and matrix function g^{ij} are real and uniformly positive on M. Then P is self-adjoint with respect to the measure $\rho(x)\,dx$, and has negative spectrum, thus we can enumerate the eigenvalues in decreasing order $\{-\lambda_j^2\}$, where $\lambda_j \to \infty$. We will call λ_j the *frequencies* of the associated eigenfunctions, as this would be exactly the frequency of the associated periodic solution to the wave equation.

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The question of interest here is to compare the L^q norm of a function f_λ, which is assumed to have only frequencies suitably close to the real number λ, to the L^2 norm of f_λ. The best possible bound on the ratio will increase as λ does, and we seek the sharp power of this growth. An eigenfunction f_λ is of course the strongest notion of being localized in frequency, but the best possible growth estimates for the L^q norm of eigenfunctions depends on subtle questions of global geometry, and is still largely an unsolved problem. We will therefore focus on the easier question of obtaining the sharp growth in λ for *spectral clusters*, which have frequencies contained in the region $[\lambda, \lambda+1]$. It turns out to be much easier to get sharp estimates in this case, since the analysis depends only on the local behavior of solutions for P.

The spectral cluster estimates can alternately be expressed in terms of $L^2 \to L^q$ bounds for the spectral projection operator Π_λ, which projects a function in $L^2(M)$ onto the span of eigenfunctions with frequencies in the range $[\lambda, \lambda+1]$.

In the case that the coefficients of P are smooth, Sogge [15] established the following bounds for the $L^2 \to L^q$ operator norm of Π_λ:

$$\big\|\Pi_\lambda f\big\|_{L^q(M)} \le C\,\lambda^{\frac{n-1}{2}(\frac12-\frac1q)}\,\|f\|_{L^2(M)}\,, \qquad 2\le q\le q_n=\tfrac{2(n+1)}{n-1}\,. \tag{1.2}$$

$$\big\|\Pi_\lambda f\big\|_{L^q(M)} \le C\,\lambda^{n(\frac12-\frac1q)-\frac12}\,\|f\|_{L^2(M)}\,, \qquad q_n\le q\le \infty\,, \tag{1.3}$$

Thus, there are two regimes to the estimates. The bounds for $2 \le q \le q_n$ can be considered as placing a lower bound on the volume of a region in which a spectral cluster can be concentrated. Precisely, if a function f is concentrated in a region of volume V, then by Holder's inequality

$$\frac{\|f\|_{L^q(M)}}{\|f\|_{L^2(M)}} \ge V^{\frac1p-\frac12}\,.$$

Consequently, the first estimate (1.2) above shows that a spectral cluster must occupy a volume of at least $\lambda^{-\frac{n-1}{2}}$.

The second estimate (1.3) is related to the Sobolev embedding theorem, and loosely says that the eigenfunctions in the range $[\lambda, 2\lambda]$ are evenly distributed over the λ subintervals of length 1.

Recently, we have been able to extend Sogge's estimates to the case of metrics which are only twice differentiable, which is the lowest level of regularity (in the Holder classes, at least) under which they can be proven, according to the counterexamples of Smith-Sogge [12].

In section 2 of these notes, we will review the proof of the above estimates for the simplest case of the flat Euclidean space. In section 3 and 4, we indicate how recent parametrix constructions for the wave equation using wave packet methods allow this proof to be extended to the setting of metrics that are only twice differentiable. We then point out in section 5 the counterexamples of Grieser and of Smith-Sogge which show that the estimates cannot be proven for $C^{1,\alpha}$ metrics. Finally, in section 6 we outline the proof that estimates (1.2) and (1.3) with a higher value for the exponent are valid for $C^{1,\alpha}$ metrics.

2. The case of $\mathbb{R}^n$

In this section we consider the simplest case where P is the standard Laplacian on the manifold $M = \mathbb{R}^n$. In this case the spectrum is continuous, and $\Pi_\lambda f$ can be interpreted as truncating the Fourier transform $\widehat{f}(\xi)$ to the set $\lambda \le |\xi| \le \lambda+1$. In the case of $\mathbb{R}^n$, these estimates are also known as *restriction estimates*, since after rescaling space by λ they can be related to multiplying $\widehat{f}$ by an approximation to surface measure on the sphere.

We start by considering examples which show that the above exponents are sharp. The example which shows that (1.2) is sharp is the simplest. Consider a Schwartz function $\phi(x)$ which has Fourier transform supported in the ball $|\xi| \le \frac{1}{2}$, intersected with the region $\xi_1 \ge 0$. Let

$$f_\lambda = e^{i\,\lambda\,x_1}\phi(x_1, \lambda^{\frac{1}{2}}x')\,,$$

where $x' = (x_2, \ldots, x_n)$. Then the Fourier transform $\widehat{f_\lambda}(\xi)$ is contained in the set $\lambda \le \xi_1 \le \lambda + \frac{1}{2}$ and $|\xi'| \le \frac{1}{2}\lambda^{\frac{1}{2}}$. This region is in turn contained in the set $\lambda \le |\xi| \le \lambda+1$. On the other hand

$$\|f_\lambda\|_{L^p} = \lambda^{-\frac{n-1}{2p}}\,\|\phi\|_{L^p}\,,$$

and comparing $p = q$ to $p = 2$ yields the sharpness of (1.2).

The example to consider for estimate (1.3) is a function f_λ where $\widehat{f_\lambda}(\xi) = 1$ on the set $\lambda \le |\xi| \le \lambda+1$, and 0 elsewhere. Then

$$\|f_\lambda\|_{L^2} \approx \lambda^{\frac{n-1}{2}}\,.$$

On the other hand, it is easy to see that $|f(x)| \gtrsim \lambda^{n-1}$ for $|x| \le \lambda^{-1}$, hence

$$\|f_\lambda\|_{L^q} \ge \lambda^{n-1-\frac{n}{q}}\,.$$

Taking the ratio shows that (1.3) cannot be improved.

We now turn to the proof of estimates (1.2) and (1.3). We begin by noting that it suffices to establish (1.3), since (1.2) follows by interpolation of the $q = q_n$ case of (1.3) with the trivial $q = 2$ bound. The estimate (1.3) is in turn proved as a consequence of certain squarefunction estimates for solutions $u(t,x)$ to the wave equation. Let

$$\|u\|_{L^q_x L^2_t([-1,1]\times\mathbb{R}^n)} = \Bigl(\int \Bigl(\int_{-1}^{1} |u(t,x)|^2\,dt\Bigr)^{\frac{q}{2}} dx\Bigr)^{\frac{1}{q}}\,.$$

Then the estimate that will yield (1.3) is

$$\|u\|_{L^q_x L^2_t([-1,1]\times\mathbb{R}^n)} \le C\,\|f\|_{H^{\delta(q)}(\mathbb{R}^n)} \tag{2.1}$$

where

$$\delta(q) = n\Bigl(\frac{1}{2} - \frac{1}{q}\Bigr) - \frac{1}{2}\,,$$

H^δ is the Sobolev space of order δ, and

$$u(t,x) = \Bigl(\exp\bigl(it\sqrt{-\Delta}\bigr)f\Bigr)(x)\,.$$

If f is a spectral cluster, then $\|f\|_{H^\delta} \approx \lambda^\delta \|f\|_{L^2}$, and u is essentially periodic in t. Multiplying $u(t,x)$ by $\exp(-it\lambda)$ and integrating over $[-1,1]$ essentially recovers f, and (1.3) follows easily from (2.1). For details, we refer to [10].

Let W denote the map

$$(Wf)(t,x) = \exp(it\sqrt{-\Delta}) f(x)\,,$$

and observe that

$$WW^*F(t,x) = \int_{-1}^{1} e^{i(t-s)\sqrt{-\Delta}} F(s,x)\,ds\,.$$

The map WW^* is easier to handle than W, since the image and domain spaces for WW^* are of the same dimension. The estimate (2.1) is equivalent to the estimate

$$\|WW^*F\|_{L^q_x L^2_t([-1,1]\times\mathbb{R}^n)} \lesssim \lambda^{n(1-\frac{2}{q})-1} \|F\|_{L^{q'}_x L^2_t([-1,1]\times\mathbb{R}^n)}\,. \tag{2.2}$$

We will now see how estimate (2.2) is established using the concentration of the Schwartz kernel of $\exp(it\sqrt{-\Delta})$ along the light cone. First, by taking a Littlewood-Paley decomposition, and a decomposition in ξ into a finite number of cones, we may assume that $\widehat{u}(t,\xi)$ is supported in the region $\lambda \le |\xi| \le 2\lambda$, and ξ near the ξ_1 axis.

We then consider the kernel of $\exp(it\sqrt{-\Delta})$ localised to frequencies in the dyadic region $[\lambda, 2\lambda]$, and also to frequencies near the ξ_1 axis,

$$K_\lambda(t,x) = \int e^{i\langle x,\xi\rangle} e^{it|\xi|}\, \beta_\lambda(\xi)\, \Gamma(\xi)\, d\xi\,,$$

where β_λ is a Littlewood-Paley cutoff to the region $\lambda \le |\xi| \le 2\lambda$, and $\Gamma(\xi)$ a conic cutoff to ξ within angle 1 of the ξ_1 axis. It is classical that, for all N,

$$\begin{aligned} |K_\lambda(t,x)| &\le C_N\, \lambda^n\, (1+\lambda|t|)^{-\frac{n-1}{2}}\, \bigl(1+\lambda\bigl||t|-|x|\bigr|\bigr)^{-N} \\ &\le C_N\, \lambda^n\, (1+\lambda|x_1|)^{-\frac{n-1}{2}}\, \bigl(1+\lambda\bigl||t|-|x|\bigr|\bigr)^{-N}\,. \end{aligned} \tag{2.3}$$

Now consider, for fixed x_1 and y_1, the operator T_{x_1,y_1} defined on functions f in the variables t, x', by the rule

$$T_{x_1,y_1} f(t,x') = \int K_\lambda(t-s, x_1-y_1, x'-y')\, f(s,y')\,ds\,dy'\,.$$

The first bound we note is that

$$\|T_{x_1,y_1} f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^2(\mathbb{R}^n)}\,. \tag{2.4}$$

This is best seen by introducing new variables $\eta_1 = |\xi|\,, \eta' = \xi'$, to write

$$T_{x_1,y_1} f(t,x') = \int e^{it\eta_1 + i\langle x',\eta'\rangle} \Bigl[e^{i(x_1-y_1)\xi_1(\eta)} \Bigl|\frac{d\xi}{d\eta}\Bigr|\, \Gamma(\xi(\eta))\beta_\lambda(\xi(\eta))\Bigr]\, \widehat{f}(\eta)\,d\eta\,,$$

and noting that the Jacobian factor is bounded on the support of $\Gamma(\xi)$, hence the term in braces is a bounded multiplier.

The next step is to observe the bound

$$\|T_{x_1,y_1} f\|_{L^\infty_{x'} L^2_t(\mathbb{R}^n)} \lesssim \lambda^{n-1}\, (1+\lambda|x_1-y_1|)^{-\frac{n-1}{2}}\, \|f\|_{L^1_{x'} L^2_t(\mathbb{R}^n)}\,. \tag{2.5}$$

This is an easy consequence of (2.3).

Interpolating (2.4) and (2.5) yields

$$\|T_{x_1,y_1}f\|_{L^q_{x'}L^2_t(\mathbb{R}^n)} \lesssim \lambda^{(n-1)(1-\frac{2}{q})}\,(1+\lambda|x_1-y_1|)^{-\frac{n-1}{2}(1-\frac{2}{q})}\|f\|_{L^{q'}_{x'}L^2_t(\mathbb{R}^n)}\,. \tag{2.6}$$

where q' is the dual index to q.

To obtain (2.2) from this, one integrates over the y_1 variable, and uses the convolution bound

$$L^{\frac{q}{2}} * L^{q'} \subset L^q\,. \tag{2.7}$$

For $q > \frac{2(n+1)}{n-1}$, the $L^{\frac{q}{2}}$ norm of the coefficient is bounded by $\lambda^{n(1-\frac{2}{q})-1}$, and the result (2.2) follows. For the critical index $q = \frac{2(n+1)}{n-1}$, one uses the result of Hardy-Littlewood that one may replace $L^{\frac{q}{2}}$ by weak-$L^{\frac{q}{2}}$ in (2.7), and the result holds there too.

3. Estimates for $C^{1,1}$ metrics

Spectral cluster estimates for variable coefficient operators P like (1.1) can be deduced from squarefunction estimates for the wave equation $\partial_t^2 - P$. The original proof of Sogge [15] proceeded differently, but the squarefunction path was followed by Mockenhaupt-Seeger-Sogge in [8]. Both of these papers considered the case that the coefficient functions of P are C^∞, in which case the calculus of Fourier integral operators is available. The Lax parametrix construction yields a representation within this calculus for the solution operator $\cos(t\sqrt{-P})$. One uses the calculus of Fourier integral operators to show that the composition WW^* is of similar type, and stationary phase shows that the kernel satisfies estimates analogous to (2.3), from which (2.1) follows.

In the case of coefficients of C^2 (or $C^{1,1}$) regularity, the asymptotic construction of Lax does not apply, and we must proceed differently. The central idea is to adapt the techniques of the paradifferential theory of Bony, by making a smooth approximation to the coefficients of P, where the approximation depends on the frequency scale at which one is working, and showing that the errors induced by modifying P can be swept under the rug for the purpose of proving the desired estimates.

To make this precise, let us suppose that u solves the Cauchy problem

$$\partial_t^2 u - Pu = 0\,, \quad u(0,x) = f(x)\,, \quad \partial_t u(0,x) = 0\,.$$

By finite propagation velocity and a partition of unity argument, we may reduce squarefunction estimates of the type (1.3) on a manifold to proving the estimate in local coordinates, and so we will assume that we are working with an operator of the form (1.1) defined globally on $\mathbb{R}^n$, with metric sufficiently close (pointwise) to the flat metric, and with coefficients belonging to $C^{1,1}(\mathbb{R}^n)$.

The first step is to use a Littlewood-Paley decomposition to reduce matters to considering the term u_λ where $\widehat{u_\lambda}(t,\xi)$ is supported in the region $|\xi| \approx \lambda$. The estimates under consideration are well behaved under the Littlewood-Paley decomposition, in that it suffices to prove them separately for each term u_λ. This step requires introducing an inhomogeneity into the equation, but it will be of the same strength as other inhomogeneities that arise shortly.

One next considers the smoothed out differential operator P_λ obtained by replacing $\mathrm{g}^{ij}(x)$ by $\mathrm{g}^{ij}_\lambda(x)$, where

$$\widehat{\mathrm{g}^{ij}_\lambda}(\xi) = \phi(\lambda^{-\frac12}|\xi|)\,\widehat{\mathrm{g}^{ij}}(\xi)\,,$$

where ϕ is a smooth cutoff to the unit ball in $\mathbb{R}^n$. Thus, g^{ij}_λ has frequencies supported in the region $|\xi| \le \lambda^{\frac12}$. We also apply this procedure to replace ρ by ρ_λ. This introduces an error term in the equation of the form

$$\Bigl[\bigl(\partial_t^2 - P\bigr) - \bigl(\partial_t^2 - P_\lambda\bigr)\Bigr]u_\lambda = (P - P_\lambda)u_\lambda\,.$$

The key estimate that controls the error is the following uniform bound:

$$\sup_x \bigl|\,\mathrm{g}^{ij}(x) - \mathrm{g}^{ij}_\lambda(x)\bigr| \lesssim \lambda^{-1}\,.$$

Since u_λ is at frequency λ, differentiation is similar to multiplying by λ. Consequently, the error term is seen to satisfy

$$\|(P - P_\lambda)u_\lambda\|_{L^\infty_t L^2_x} \le C\,\lambda\,\|f_\lambda\|_{L^2}\,.$$

If we consider $(P - P_\lambda)u_\lambda$ as an inhomogeneity, then this shows that the inhomogeneity is of the same strength as the initial data f_λ, and thus we can focus on proving estimates for solutions to the operator $\partial_t^2 - P_\lambda$.

It is more convenient to work with a first-order equation, which may be attained by factoring

$$\partial_t^2 - P_\lambda = -\bigl(D_t - p_\lambda(x, D_x)\bigr)\bigl(D_t + p_\lambda(x, D_x)\bigr) + r_\lambda(x, D_x)\,,$$

where $D_t = -i\partial_t$, and p_λ is a pseudodifferential operator with symbol

$$p_\lambda(x,\xi) = \Bigl(\,\sum_{i,j=1}^n \mathrm{g}^{ij}_\lambda(x)\,\xi_i\,\xi_j\,\Bigr)^{\frac12}\,.$$

The term r_λ is a pseudodifferential operator of order 0, and $r_\lambda u_\lambda$ may be absorbed into the driving term.

We can split u_λ into two pieces, corresponding to positive and negative t frequencies, each of which is contained in a region where one of the factors is elliptic. Restricting attention to one term, we may thus reduce consideration to the equation

$$D_t u_\lambda + p_\lambda(x, D_x)u_\lambda = F_\lambda\,, \qquad u_\lambda(0,x) = f_\lambda(x)\,,$$

and we seek to prove

$$\|u_\lambda\|_{L^q_x L^2_t([-1,1]\times\mathbb{R}^n)} \le C\,\lambda^{\delta(q)}\bigl(\,\|f_\lambda\|_{L^2_x(\mathbb{R}^n)} + \|F_\lambda\|_{L^1_t L^2_x([-1,1]\times\mathbb{R}^n)}\bigr)\,. \tag{3.1}$$

We could replace L^1_t by L^∞_t in the norm for F_λ, but L^1_t is the norm which is natural for the energy norm. It is also convenient to assume that the x-Fourier transform of the symbol p_λ is localized to frequencies at most $\lambda^{\frac12}$, which can be arranged since the error can be absorbed into F_λ.

The original problem, that of proving estimates for an operator P with only twice differentiable coefficients, has thus been reduced to proving the same estimates for the smoothed out operator P_λ, for solutions at frequency scale λ. The parametrix construction of Lax is still not helpful, since the symbol of P_λ is essentially of class $S_{\frac12,\frac12}$, where the construction does not yield an asymptotically convergent expansion. In the next section we review the method of using wave packet techniques to construct approximate (but not asymptotic) parametrices for $D_t + p_\lambda$, and to reduce proving estimates to a question of stationary phase.

4. Wave packet methods

A very powerful tool for studying solutions to hyperbolic equations, and which we use to establish (3.1), is the Cordoba-Fefferman wave packet transform [3]. This has since been generalized in what has come to be known as the FBI transform, see [5]. The idea is to decompose $u_\lambda(t,\,\cdot\,)$ into a continuous superposition of packets localized in phase space, on each of which the action of $p_\lambda(x,D_x)$ is well approximated by the infinitesimal flow along the Hamiltonian curves of p_λ. The coefficient function of u_λ in this continuous frame will then be seen to satisfy a simple first order ODE, which can be exactly integrated.

In order to maintain the compact support of $\widehat{u_\lambda}(t,\xi)$, we use as building block for our transform a Schwartz function g whose Fourier transform is supported in the unit ball of $\mathbb{R}^n$, and for which

$$\|g\|_{L^2} = (2\pi)^{-\frac{n}{2}}\,.$$

We then define

$$\bigl(T_\lambda f\bigr)(x,\xi) = \lambda^{\frac{n}{4}} \int e^{-i\langle \xi, z-x\rangle}\, g\bigl(\lambda^{\frac12}(z-x)\bigr)\, f(z)\, dz\,.$$

The map T_λ is seen to be a continuous tight-frame transform, in the sense that

$$f(y) = \lambda^{\frac{n}{4}} \int e^{i\langle \xi, y-x\rangle}\, g\bigl(\lambda^{\frac12}(y-x)\bigr)\,\bigl(T_\lambda f\bigr)(x,\xi)\, dx\, d\xi\,,$$

so that $T_\lambda^* T_\lambda = I$. In particular,

$$\|T_\lambda f\|_{L^2(\mathbb{R}^{2n}_{x,\xi})} = \|f\|_{L^2(\mathbb{R}^n_x)}\,.$$

The wave-packet transform T_λ thus decomposes a function into a continuous superposition of packets, with the packet associated to the phase-space point (x,ξ) localized in Fourier transform to the ball of radius $\lambda^{\frac12}$ centered at ξ, and concentrated (in the sense of rapid decay) in the ball of radius $\lambda^{-\frac12}$ about the point x. We can think of this wave packet as localized to a product ball, which has symplectic volume 1.

The important property of this product ball in phase space is that it is the largest set on which the symbol p_λ is equal to its first order Taylor expansion within a bounded error. Precisely, we may write

$$p_\lambda(y,\eta) = d_\xi p_\lambda(x,\xi)\cdot\eta + d_x p_\lambda(x,\xi)\cdot(y-x) + r_{x,\xi}(y,\eta)\,,$$

where $|r_{x,\xi}(y,\eta)| \lesssim 1$ on the product ball. In fact, the error $r_{x,\xi}(y,\eta)$ is seen to behave as a symbol of type $(\frac12,\frac12)$ of order 0, with bounds that grow polynomially in $y-x$ at the correct scale. Consequently, we may write

$$\Bigl(p_\lambda(y,D_y) - i d_\xi p_\lambda(x,\xi)\cdot d_x + i d_x p_\lambda(x,\xi)\cdot d_\xi\Bigr)\Bigl[e^{i\langle\xi,y-x\rangle}\, g\bigl(\lambda^{\frac12}(y-x)\bigr)\Bigr] \\ = e^{i\langle\xi,y-x\rangle}\, g_{x,\xi}\bigl(\lambda^{\frac12}(y-x)\bigr)$$

where $g_{x,\xi}$ denotes a Schwartz function depending on the parameters (x,ξ), but which belongs to a bounded family of Schwartz functions as (x,ξ) vary. If we define R_λ similar to T_λ but using the (x,ξ) dependent family $g_{x,\xi}$, then we have

$$T_\lambda\big(p_\lambda(y,D_y)f\big)(x,\xi) = i\big(d_\xi p_\lambda(x,\xi)\cdot d_x - d_x p_\lambda(x,\xi)\cdot d_\xi\big)T_\lambda f(x,\xi) + R_\lambda f(x,\xi)\,.$$

We now define

$$\tilde u_\lambda(t,x,\xi) = \big(T_\lambda u_\lambda(t,\,\cdot\,)\big)(x,\xi)\,.$$

Then the error term $R_\lambda u_\lambda$ is of the same strength as $T_\lambda F_\lambda$, and we may combine them to write

$$\big(\partial_t - d_\xi p_\lambda(x,\xi)\cdot d_x + d_x p_\lambda(x,\xi)\cdot d_\xi\big)\tilde u_\lambda(t,x,\xi) = \tilde F_\lambda(t,x,\xi)\,,\quad \tilde u_\lambda(0,x,\xi) = \tilde f_\lambda(x,\xi)\,.$$

This is now an ordinary differential equation which can be solved to express $\tilde u_\lambda$ as an integral over the hamiltonian flow of $\tilde F_\lambda$. We emphasize, though, that the term $\tilde F_\lambda$ contains various error terms expressed in terms of u_λ, which we can control in the L^2 norm by energy conservation estimates on u_λ. So while this does not give a method for solving the wave equation, it is useful for proving estimates of Strichartz type or squarefunction type, in which one seeks to control a mixed L^p norm of u in terms of a Sobolev norm on the initial data of u. Indeed, the solution of the above equation expresses u as an integrable superposition over s of solutions to the homogeneous equation restricted to $t>s$. For L^p estimates the integral over s can be taken outside the integral, and we are reduced to considering the case $\tilde F_\lambda\equiv 0$.

We define a one-parameter group action on $L^2_{x,\xi}(\mathbb{R}^{2n})$ by setting

$$\big(S(t)f\big)(x,\xi) = f(\chi_t^{-1}(x,\xi))\,,$$

where χ_t is the diffeomorphism group generated by the hamiltonian flow along $p_\lambda(x,\xi)$. Then the solution to the homogeneus ODE with data $\tilde f_\lambda$ is given by

$$\tilde u_\lambda(t,x,\xi) = \big(\,S(t)\tilde f_\lambda\,\big)(x,\xi)\,.$$

Note that $S(t)$ is a unitary group since the hamiltonian flow preserves the measure $dx\,d\xi$. Thus, the squarefunction estimate (3.1) is reduced to showing

$$\|W_\lambda f_\lambda\|_{L^q_x L^2_t([-1,1]\times\mathbb{R}^n)} \lesssim \|f_\lambda\|_{L^2(\mathbb{R}^{2n}_{x,\xi})}\,,$$

where $W_\lambda f_\lambda(t,x) = T^*_\lambda S(t) f_\lambda$. This estimate in turn is implied by the estimate

$$\|W_\lambda W_\lambda^* F\|_{L^q_x L^2_t([-1,1]\times\mathbb{R}^n)} \lesssim \|F\|_{L^{q'}_x L^2_t([-1,1]\times\mathbb{R}^n)}\,,$$

where the operator $W_\lambda W_\lambda *$ takes the form

$$\big(W_\lambda W_\lambda^* F\big)(t,x) = \int_{-1}^{1}\big(T^*_\lambda S(t-s)T_\lambda F(s,\,\cdot\,)\big)(x)\,ds\,,$$

and we may assume that everything is localized to $|\xi|\approx\lambda$.

The operator $T^*_\lambda S(t-s)T_\lambda$ is the idealized model for $e^{i(t-s)\sqrt{-P}}$, localized to the dyadic frequency scale λ, in which one makes the approximation that a Cordoba-Fefferman wave packet simply gets translated for time $t-s$ along the hamiltonian flow through its center. It is clear that the integral kernel of this operator will be concentrated along the light cone, and indeed a careful analysis (see [10] for details) shows that the associated kernel satisfies

$$|K_\lambda(t-s,x,y)| \lesssim \lambda^n\,\big(\,1+\lambda\,|t-s|\big)^{-\frac{n-1}{2}}\,\big(1+\lambda\,|\,|t-s|-\Phi_y(x)|\,\big)^{-N}\,,$$

where $|t| = \Phi_y(x)$ defines the light cone centered at the point $(0, y)$. As in the flat case, these estimates imply the squarefunction estimates. They also yield a proof of the usual family of Strichartz estimates, which were established for $n = 2, 3$ in [9] and in all dimensions by Tataru in [17]. We remark that the wave packet techniques mentioned here are essentially those used by Tataru in [17]. The paper [9] used a discrete frame of wave packets based upon the second dyadic decomposition (see [16]).

5. Manifolds with boundary and Lipschitz metrics

In this section, we observe how the spectral cluster estimates (1.2) can fail if M is replaced by a manifold with boundary, and the eigenfunctions are required to satisfy Dirichlet (or Neumann) conditions on ∂M. This fact was first observed in the thesis of Grieser [7], who produced the example we exhibit here. The geometry of the boundary ∂M plays a crucial role. If the boundary is geodesically convex, in the sense that nearly-tangent geodesics originating from ∂M intersect ∂M again in short time, then the estimates fail. If the boundary is strictly geodesically concave, so that even tangent geodesics starting in ∂M move away from ∂M, then it was shown in [13] that the estimates (1.2) and (1.3) do hold. The question of whether the estimates hold in case of (not necessarily strict) convexity remains open, but we anticipate that they do. It is also expected that estimate (1.3) holds for q sufficiently large.

The counterexample for manifolds with boundary immediately yields an example of an operator of type (1.1) on a boundary-free manifold, where the coefficients are of Lipschitz regularity, for which the estimate (1.2) fails. The construction involves joining two copies of M together along ∂M, and extending the metric across ∂M in an even manner with respect to geodesic normal coordinates. The extended metric has a simple corner type singularity across ∂M. If an eigenfunction f is extended in an odd manner across ∂M (or an even manner in case of Neumann conditions) then the extended function is an eigenfunction on the doubled manifold.

This procedure shows that any estimate for operators with Lipschitz coefficients directly yields an estimate for manifolds with boundary. Of course, the class of manifolds with boundary leads to singularities of a very special nature, and one expects in general better estimates to hold. However, for $n = 2$ and q below the critical power as in (1.2), it turns out that the best possible estimates coincide for the two cases.

The example of Grieser is simple: take $M \subset \mathbb{R}^2$ to be the unit disc $D = \{x : |x| \leq 1\}$ with the usual Laplacian, and impose Dirichlet conditions at $|x| = 1$. Then the eigenfunctions can be expressed in polar coordinates (r, θ) as

$$f(r, \theta) = e^{in\theta} J(cr) \,,$$

where J is an appropriate Bessel function and c is a zero of J. If one takes c to be the first zero of J, then f is an eigenfunction of frequency $\lambda \approx n$, and the asymptotics of Bessel functions shows that f is highly concentrated in the set $1 - n^{-\frac{2}{3}} \leq r \leq 1$. This set has volume $\lambda^{-\frac{2}{3}}$ which violates the inequality (1.2), since (1.2) would imply that f can be concentrated in a set of volume no less than $\lambda^{-\frac{1}{2}}$.

This concentration behavior can be exhibited without appealing to Bessel functions by looking at a closely related model. If we introduce coordinates $y = 1 - r$, $x = \theta$, then the top order terms in the Laplace are

$$\frac{\partial^2}{\partial_y^2} + \frac{1}{(1-y)^2}\frac{\partial^2}{\partial_x^2}\,.$$

A model which has similar geodesic behavior near $y = 0$, and which is reflected in an even manner about $y = 0$, is

$$\frac{1}{1-|y|}\Bigl(\frac{\partial^2}{\partial_y^2} + \frac{\partial^2}{\partial_x^2}\Bigr)\,.$$

If we look for eigenfunctions of the form $\exp(ix\xi)f_\xi(y)$, with eigenvalue $-\lambda_\xi^2$, then we can rewrite the eigenvalue condition as

$$-\partial_y^2 f_\xi(y) + \lambda(\xi)^2\,|y| f_\xi(y) = \bigl(\lambda_\xi^2 - \xi^2\bigr)\, f_\xi(y)\,. \tag{5.1}$$

This equation is just a rescaled version of the equation

$$-\partial_y^2 A(y) + |y|\,A(y) = c\,A(y)\,,$$

which admits an even solution $A(y) = Ai(|y| - c)$, with Ai the unique bounded solution to the Airy equation, and we require $Ai'(-c) = 0$. Precisely, the function

$$f_\xi(y) = A(\lambda_\xi^{\frac{2}{3}}\,y)$$

satisfies (5.1) provided $\lambda_\xi^2 - \xi^2 = c\,\lambda_\xi^{\frac{4}{3}}$, which means that

$$\lambda_\xi = \xi + \frac{c}{2}\,\xi^{\frac{1}{3}} + r(\xi)\,,$$

with $r(\xi)$ bounded. Since the Airy function decreases exponentially for $y > 0$, it follows that $f_\xi(y)$ is localized exponentially in a $\lambda^{-\frac{2}{3}}$ neighborhood of $y = 0$.

This model was generalized in [12] to produce $C^{1,\alpha}$ metrics in higher dimensions by taking $y \in \mathbb{R}^{n-1}$, and considering the metric

$$\frac{1}{1-|y|^{1+\alpha}}\Bigl(\Delta_y + \frac{\partial^2}{\partial_x^2}\Bigr)\,.$$

Similar steps lead to eigenfunctions

$$e^{ix\xi}A_\alpha(\lambda_\xi^\delta y)\,,$$

where

$$\lambda_\xi^2 - \xi^2 = c_\alpha\,\lambda_\xi^{2\delta}\,, \qquad \delta = \frac{2}{3+\alpha}\,,$$

and A_α satisfies

$$-\Delta A_\alpha(y) + |y|^{1+\alpha}\,A_\alpha(y) = c_\alpha\,A_\alpha(y)\,.$$

By taking A_α to be the ground state of this Schrodinger equation, we obtain a radial, exponentially decreasing solution. The result is that $C^{1,\alpha}$ metrics can have eigenfunctions of frequency λ which are concentrated in a tube of radius $\lambda^{-\delta}$. For $\alpha < 1$, then $\delta > \frac{1}{2}$, and this contradicts the estimate (1.2).

It is interesting to ask whether these examples also violate the Strichartz estimates. It turns out, as noted in [12], that they produce somewhat weaker than expected violations, and only in dimension $n \geq 3$, due to the longitudinal dispersion that comes from the eigenvalue relation

$$\lambda_\xi = \xi + \frac{c_\alpha}{2}\,\xi^{2\delta-1} + r(\xi)\,.$$

Modified examples of $C^{1,\alpha}$ metrics were produced which eliminate this longitudinal dispersion, and yield counterexamples to the Strichartz estimates that do coincide with the positive results established by Tataru [17]. For boundary value problems, however, the longitudinal dispersion is present, and it remains an open question whether, for example, the Strichartz estimates hold on the unit disc with Dirichlet condtions.

It also remains open whether (1.3), which is a consequence of the squarefunction estimate (2.1), holds for Lipschitz or $C^{1,\alpha}$ metrics for some range of q. The above examples violate (1.3) only for q near q_n, but not for large q. Very recent work of Smith and Sogge shows that for $n = 2$ and manifolds with boundary, estimate (1.3) does indeed hold for $q > 8$. This is the largest possible range on which (1.3) can hold by the counterexample of Grieser.

In the next section, we illustrate how to establish a weaker version of (1.3) in the case of Lipschitz metrics which, while not known to be sharp, does yield the best possible version of (1.2) after interpolation with the trivial $q = 2$ endpoint.

6. Lipschitz metrics, positive results

The examples of the previous section constructed Lipschitz metrics and functions f for which

$$\|\Pi_\lambda f\|_{L^q} \approx \lambda^{\frac{2}{3}(n-1)(\frac{1}{2}-\frac{1}{q})} \, \|f\|_{L^2} \,. \tag{6.1}$$

The examples were only constructed on an open set, but one can simply cut them off to $|y| \le \frac{1}{2}$ with error exponentially small in λ, and (6.1) still holds.

For $q = q_n$, we note that

$$\frac{2}{3}(n-1)\Big(\frac{1}{2} - \frac{1}{q_n}\Big) = n\Big(\frac{1}{2} - \frac{1}{q_n}\Big) - \frac{1}{2} + \frac{1}{3q_n}\,.$$

Thus, at the critical index $q = q_n$ estimate (1.3) experiences an increase of $\frac{1}{3q}$ in the exponent. This can alternately be expressed by saying that estimate (2.1) holds (at best) with a loss of $\frac{1}{3q}$ derivatives at $q = q_n$.

For Strichartz estimates, which take the form

$$\|u\|_{L^q_t L^p_x([-1,1]\times\mathbb{R}^n)} \lesssim \|u\|_{L^\infty_t H^{s(q,p)}_x([-1,1]\times\mathbb{R}^n)} + \|F\|_{L^1_t H^{s(q,p)-1}_x([-1,1]\times\mathbb{R}^n)}\,, \tag{6.2}$$

for solutions to

$$\partial_t^2 u - Pu = F\,,$$

the estimates were shown by Tataru [17] to hold for Lipschitz metrics with a loss of $\frac{1}{3q}$ derivatives in the index $s(q,p)$ (relative to smooth or C^2 metrics). This result is sharp (for critical values of q, p) by the examples of Smith-Tataru [14].

Similar ideas show that the squarefunction estimates (2.1) hold with loss of $\frac{1}{3q}$ derivatives. Only the case $q = q_n$ is "critical", in that the estimates for $q > q_n$ follow from the case $q = q_n$ together with Sobolev embedding. Consequently the counterexamples only yield sharpness of this result for $q = q_n$.

We illustrate here the key idea of the proof, which is surprisingly simple: after localization of u in frequency to a given frequency scale λ, and suitable λ-dependent frequency localization of the coefficients, one makes a λ-dependent dilation of space time to reduce matters to the case of C^2 metrics. The factor of $\frac{1}{3q}$ arises from balancing the demands of minimizing the size of errors against the demand of making the frequency-truncated coefficients as smooth as needed.

For simplicity, we consider the case of the Strichartz estimates (6.2), since the arguments are simpler. We will then mention the necessary modifications needed to handle the squarefunction estimates (2.1).

The starting point is the following:

> The estimate (6.2) holds with constant depending only on the C^2 norm of the metric (and the dimension).

(We also implicitly assume here that P is sufficiently close to the flat metric, so as to assure that it is uniformly elliptic.)

One begins by taking a Littlewood-Paley expansion to assume that u is localized to frequency scale λ. Next, one smooths the metric coefficients g^{ij} and ρ by truncating their frequencies. The appropriate localization for Lipschitz metrics is to let

$$\widehat{\mathrm{g}_\lambda^{ij}}(\xi) = \phi(\lambda^{-\frac{2}{3}}\xi)\,\widehat{\mathrm{g}^{ij}}(\xi)\,,$$

and similarly for ρ. Since $\mathrm{g}^{ij}(x)$ is Lipschitz, it holds that

$$\sup_x \bigl|\,\mathrm{g}_\lambda^{ij}(x) - \mathrm{g}^{ij}(x)\bigr| \lesssim \lambda^{-\frac{2}{3}}\,,$$

with the result that

$$\|(P-P_\lambda)u_\lambda\|_{L^\infty_t L^2_x([-1,1]\times\mathbb{R}^n)} \lesssim \lambda^{2-\frac{2}{3}}\,\|u_\lambda\|_{L^\infty_t L^2_x([-1,1]\times\mathbb{R}^n)}\,.$$

This is off by a factor of $\lambda^{\frac{1}{3}}$ from what would allow one to absorb this error into the driving term F. On the other hand, if I is any interval of length $\lambda^{-\frac{1}{3}}$ contained in $[-1,1]$, then

$$\|(P-P_\lambda)u_\lambda\|_{L^1_t L^2_x(I\times\mathbb{R}^n)} \lesssim \lambda\,\|u_\lambda\|_{L^\infty_t L^2_x([-1,1]\times\mathbb{R}^n)}\,.$$

The next step is to dilate space-time by a factor of $\lambda^{\frac{1}{3}}$. Thus, we consider the scaled operator $\tilde P_\lambda$ with coefficients $\mathrm{g}_\lambda^{ij}(\lambda^{-\frac{1}{3}}x)$. These rescaled coefficients satisfy

$$\sup_x \bigl|D^2\mathrm{g}_\lambda^{ij}(\lambda^{-\frac{1}{3}}x)\bigr| = (\lambda^{-\frac{1}{3}})^2\,\sup_x\bigl|D^2\mathrm{g}_\lambda^{ij}(x)\bigr| \lesssim 1\,,$$

the second step by the frequency localization and the fact that g^{ij} is Lipschitz. Consequently the rescaled metric is C^2, with bounds independent of λ. By assumption, one can prove the Strichartz estimates (with no loss) for the operator $\partial_t^2 - \tilde P_\lambda$ on time intervals of length 1.

The Strichartz estimates are dilation invariant, however, in that the norms on both sides of (6.2) scale by the same factor under dilation. (Strictly speaking this requires the use of homogeneous Sobolev spaces, but since we are localized at frequency larger than

1 this is not an issue.) Putting this together, we have, for any slice $S = I \times \mathbb{R}^n$ with I of length $\lambda^{-\frac{1}{3}}$,

$$\begin{aligned}\|u_\lambda\|_{L^q_t L^p_x(S)} &\lesssim \|u_\lambda\|_{L^\infty_t H^{s(q,p)}_x(S)} + \|(P - P_\lambda)u_\lambda\|_{L^1_t H^{s(q,p)-1}_x(S)} + \|F_\lambda\|_{L^1_t H^{s(q,p)-1}_x(S)} \\ &\lesssim \|u_\lambda\|_{L^\infty_t H^{s(q,p)}_x([-1,1]\times\mathbb{R}^n)} + \|F_\lambda\|_{L^1_t H^{s(q,p)-1}_x([-1,1]\times\mathbb{R}^n)} \,.\end{aligned}$$

Adding up over the $\lambda^{\frac{1}{3}}$ sets S that make up $[-1,1] \times \mathbb{R}^n$ yields

$$\|u_\lambda\|_{L^q_t L^p_x([-1,1]\times\mathbb{R}^n)} \lesssim \lambda^{\frac{1}{3q}} \Big(\|u_\lambda\|_{L^\infty_t H^{s(q,p)}_x([-1,1]\times\mathbb{R}^n)} + \|F_\lambda\|_{L^1_t H^{s(q,p)-1}_x([-1,1]\times\mathbb{R}^n)} \Big) ,$$

which equates to a loss of $\frac{1}{3q}$ derivatives in the Sobolev index.

For the squarefunction estimates (2.1), the proof is similar, but the role of time and spatial variables is reversed. In essence, after localizing $\widehat{u}$ to a cone near the ξ_1 axis, one can consider P as hyperbolic in the x_1 variable, and interchange the roles of x_1 and t. Similar steps show that the squarefunction estimates hold for u_λ with no loss on slices of thickness $\lambda^{-\frac{1}{3}}$ in the x_1 direction, and adding up over such slices yields a loss of $\lambda^{\frac{1}{3q}}$.

There is also an appropriate truncation and dilation argument for $C^{1,\alpha}$ metrics, which gives best possible results for both Strichartz and squarefunction estimates. Precisely, the result is that for such metrics, there is a loss of $\frac{\sigma}{q}$ derivatives relative to the C^2 index, where

$$\sigma = \frac{1-\alpha}{3+\alpha} \,.$$

The case of Strichartz was established by Tataru [17], who also considered time dependendent metrics. The case of squarefunction estimates is established in [11]. The correct amount of rescaling is dictated by the counterexamples of Smith and Sogge [12], as seen, for example, in the cubic nature of the rescaling in the Lipschitz case we have considered above. Results yielding nonsharp versions of Strichartz for metrics less regular than C^2 had been obtained by Bahouri and Chemin [1], using a different scaling procedure.

7. Open questions

The question of spectral cluster estimates for metrics less regular than Lipschitz is still largely open. An example of Davies [4] shows that for metrics which are merely bounded, one cannot in general prove any estimate stronger than that given by Sobolev embedding. This example is closely related to localization phenomena for highly oscillatory metrics, and the failure of energy-flux bounds. Indeed, below Lipschitz a key step in the proof of the spectral cluster bounds fails: the wave evolution operator restricted from a slice $x_1 = const$ to another such slice need not be bounded in L^2. For examples see Castro and Zuazua [2]. Tataru [17] showed that for Strichartz estimates one can extend the rescaling arguments to C^α metrics for $0 < \alpha < 1$ but one needs, in effect, to assume conservation of energy. (For the time-independent metrics we consider in these notes this is satisfied.) The absence of energy-flux (sidewise energy conservation) for C^α metrics is the reason such arguments do not work for squarefunction estimates. The currently known examples of localized eigenfunctions for C^α metrics however do not rule out the possibility of some results in this setting which improve upon Sobolev embedding.

A promising avenue of research is the analysis of boundary value problems by the reflection method noted in section 5. This procedure has already yielded results in two dimensions for spectral cluster estimates, in ongoing joint work with Sogge. It would be interesting to apply this method to study the wave equation on a domain exterior to a convex obstacle, where convexity implies that the counterexample noted by Grieser does not occur. It remains to effectively combine the short-time, near-boundary treatment of the obstacle problem as a Lipschitz metric with the long time treatment of the wave equation through scattering methods.

There also remains the question of gathering sharper information on the actual fundamental solution of the wave equation for metrics of limited differentiability. In the work cited in this review, all results proceed by approximating the wave operator by a smoothed out version and showing that, for the purpose of proving space-time L^p estimates, the induced errors can be absorbed in the driving term. Recently Geba and Tataru [6] have used a finer multi-scale resolution of the wave equation to establish the fixed time *dispersive estimates* for the wave group. It still remains to quantify to what degree the fundamental solution is concentrated along the light cone for C^2 metrics, and whether any significant statements can be made for metrics less regular than C^2.

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Department of Mathematics, University of Washington, Seattle, WA 98195

E-mail address: hart@math.washington.edu