Lecture 15: $f^\sharp$ and interpolation

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Definition

For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define

$$f^\sharp(x) = \sup_{x \in Q} \frac{1}{m(Q)} \int_Q |f(y) - f_Q| \, dy$$

Remark: we showed earlier that

$$\tfrac{1}{2} f^\sharp(x) \le \sup_{x \in Q} \frac{1}{m(Q)} \left(\inf_{c \in \mathbb{C}} \int_Q |f(y) - c| \, dy \right) \le f^\sharp(x) \, .$$

- $\|f\|_{\mathrm{BMO}} = \|f^\sharp\|_{L^\infty}$.
- $f^\sharp(x) \le 2Mf(x)$ (M = off center cube maximal function) so

$$\|f^\sharp\|_{L^p(\mathbb{R}^n)} \le C_{n,p} \, \|f\|_{L^p(\mathbb{R}^n)} \, , \qquad 1 < p \le \infty \, .$$

dyadic cube = cube of sidelength 2^j, vertices $\in 2^j\,\mathbb{Z}^n$, $j \in \mathbb{Z}$.

Definition: given $f \in L^p(\mathbb{R}^n)$, some $1 \le p < \infty$, and $\alpha > 0$

Let $\mathcal{Q}_\alpha(f)$ = collection of dyadic cubes Q such that

$$\frac{1}{m(Q)} \int_Q |f(y)|\, dy > \alpha \quad \text{and} \quad \frac{1}{m(Q')} \int_{Q'} |f(y)|\, dy \le \alpha$$

for all dyadic Q' that contain Q. (So cubes in $\mathcal{Q}_\alpha(f)$ are disjoint.)

- For $Q \in \mathcal{Q}_\alpha(f)$: $\quad \alpha < \dfrac{1}{m(Q)} \displaystyle\int_Q |f(y)|\, dy \le 2^n \alpha\,.$
- $|f(x)| \le \alpha$ a.e. on $\mathbb{R}^n \setminus \cup_{\mathcal{Q}_\alpha(f)} Q$, since the dyadic cubes containing a point x shrink nicely to x.
- $\cup_{\mathcal{Q}_\alpha(f)} Q \subset \{x : Mf(x) > \alpha\}$, where M is off-center maximal.
- There is an upper bound on the diameter of cubes in $\mathcal{Q}_\alpha(f)$.

Lemma 1

With $c_1 = 1 + 6^n$, we have $Mf(x) \leq c_1\alpha$ for $x \notin \cup_{\mathcal{Q}_\alpha(f)} Q^*$.

Therefore: $m(\{x : Mf(x) > c_1\alpha\}) \leq 3^n \sum_{\mathcal{Q}_\alpha(f)} m(Q)$.

Proof. Suppose $x \notin \cup_{\mathcal{Q}_\alpha(f)} Q^*$ and $Q' \ni x$, Q' any cube.

$$\int_{Q'} |f(y)|\, dy = \int_{Q' \cap (\cup_{\mathcal{Q}_\alpha(f)} Q)} |f(y)|\, dy + \int_{Q' \cap (\cup_{\mathcal{Q}_\alpha(f)} Q)^c} |f(y)|\, dy$$

- Second term $\leq \alpha\, m(Q')$, since $|f(y)| \leq \alpha$ for $y \notin \cup_{\mathcal{Q}_\alpha(f)} Q$.
- $Q' \not\subset Q^*$ when $Q \in \mathcal{Q}_\alpha(f)$, since x is not, so for $Q \in \mathcal{Q}_\alpha(f)$:

$$Q' \cap Q \neq \emptyset \quad \Rightarrow \quad \text{diam}(Q') > \text{diam}(Q) \quad \Rightarrow \quad Q \subset Q'^*$$

$$\text{First term} \leq \sum_{Q \in \mathcal{Q}_\alpha(f) : Q \subset Q'^*} \int_Q |f(y)|\, dy$$

$$\leq \sum_{Q \in \mathcal{Q}_\alpha(f) : Q \subset Q'^*} 2^n \alpha\, m(Q) \leq 6^n \alpha\, m(Q')$$

Lemma 2

Let $\mu(\alpha) = \sum_{Q \in \mathcal{Q}_\alpha(f)} m(Q)$. Then for all α, $c > 0$:

$$\mu(\alpha) \leq m\big(\{x : f^\sharp(x) > c\alpha\}\big) + 2c\,\mu(2^{-n-1}\alpha)\,.$$

Proof. If $Q \in \mathcal{Q}_\alpha(f)$, then $Q \subset Q_0$ for some $Q_0 \in \mathcal{Q}_{2^{-n-1}\alpha}(f)$. So it suffices to show that, for any $Q_0 \in \mathcal{Q}_{2^{-n-1}\alpha}(f)$,

$$\sum_{Q \in \mathcal{Q}_\alpha(f) : Q \subset Q_0} m(Q) \;\leq\; m\big(\{x \in Q_0 : f^\sharp(x) > c\alpha\}\big) \;+\; 2c\,m(Q_0)\,.$$

Case 1: $\quad \dfrac{1}{m(Q_0)} \displaystyle\int_{Q_0} |f(y) - f_{Q_0}|\,dy \;>\; c\alpha\,.$

Then $f^\sharp(x) > c\alpha$ on Q_0, so

$$m\big(\{x \in Q_0 : f^\sharp(x) > c\alpha\}\big) \;=\; m(Q_0) \;\geq \sum_{Q \in \mathcal{Q}_\alpha(f) : Q \subset Q_0} m(Q)\,.$$

Case 2: $\dfrac{1}{m(Q_0)}\displaystyle\int_{Q_0}|f(y)-f_{Q_0}|\,dy \;\le\; c\alpha\,.$ We will show

$$\sum_{Q\in\mathcal{Q}_\alpha(f):Q\subset Q_0} m(Q) \;\le\; 2c\,m(Q_0)\,.$$

- Since $Q_0\in\mathcal{Q}_{2^{-n-1}\alpha}(f)\,,$ we have $|f_{Q_0}|\le 2^n\cdot 2^{-n-1}\alpha \;\le\; \frac{1}{2}\alpha\,.$
- For $Q\in\mathcal{Q}_\alpha(f)\,,$ we have $\displaystyle\int_Q|f(y)|\,dy>\alpha m(Q)\,,$ so

$$\int_Q|f(y)-f_{Q_0}|\,dy>\tfrac{1}{2}\alpha\,m(Q)\,.$$

$$\begin{aligned}\tfrac{1}{2}\alpha\sum_{Q\in\mathcal{Q}_\alpha(f):Q\subset Q_0} m(Q) \;&\le\; \sum_{Q\in\mathcal{Q}_\alpha(f):Q\subset Q_0}\int_Q|f(y)-f_{Q_0}|\,dy\\ &\le\; \int_{Q_0}|f(y)-f_{Q_0}|\,dy\\ &\le\; c\alpha\,m(Q_0)\end{aligned}$$

Theorem: assume that $1 \le p_0 < p < \infty$.

If $f \in L^{p_0}(\mathbb{R}^n)$, then $\|f\|_{L^p} \le C_p \|f^\sharp\|_{L^p}$.

Proof. Lemma 1 says that $\lambda_{Mf}(c_1\alpha) \le 3^n\mu(\alpha)$, hence

$$\|f\|_{L^p} \le \|Mf\|_{L^p} \le C_{p,n} \int_0^\infty \alpha^{p-1}\,\mu(\alpha)\,d\alpha\,.$$

Lemma 2 says $\mu(\alpha) \le \lambda_{f^\sharp}(c\alpha) + 2c\mu(2^{-n-1}\alpha)$, so for $N < \infty$

$$\int_0^N \alpha^{p-1}\,\mu(\alpha)\,d\alpha \le \int_0^N \alpha^{p-1}\,\lambda_{f^\sharp}(c\alpha)\,d\alpha + c\,2^{1+(n-1)p}\int_0^{2^{-n-1}N} \alpha^{p-1}\mu(\alpha)\,d\alpha$$

$f \in L^{p_0}(\mathbb{R}^n) \Rightarrow \mu(\alpha) \le C\,\alpha^{-p_0}$ (recall that $\mu(\alpha) \le \lambda_{Mf}(\alpha)$)

so integrals with $\mu(\alpha)$ are finite, and taking c small this implies

$$\int_0^N \alpha^{p-1}\,\mu(\alpha)\,d\alpha \le C_{n,p} \int_0^N \alpha^{p-1}\,\lambda_{f^\sharp}(c\alpha)\,d\alpha$$

Let $N \to \infty$ to get $\|f\|_{L^p} \le C_{n,p} \|f^\sharp\|_{L^p}$. $\square$

Theorem: assume $1 \leq p_0 < \infty$, and T is linear operator on L^{p_0}.

If for all $f \in L^{p_0}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ the following hold

$$\|Tf\|_{L^{p_0}} \leq C\,\|f\|_{L^{p_0}}\,, \qquad \|Tf\|_{\mathrm{BMO}} \leq C\,\|f\|_{L^\infty}\,,$$

then for all $p_0 < p < \infty$ we have $\|Tf\|_{L^p} \leq C_p\,\|f\|_{L^p}$.

Proof. Consider the operator $f \to (Tf)^\sharp$:

- $(Tf)^\sharp$ is sub-additive, since $(Tf + Tg)^\sharp \leq (Tf)^\sharp + (Tg)^\sharp$.
- $\|(Tf)^\sharp\|_{L^{p_0,\infty}} \leq C\,\|f\|_{L^{p_0}}\,, \qquad \|(Tf)^\sharp\|_{L^\infty} \leq C\,\|f\|_{L^\infty}$.
- By previous Theorem and Marcienkiewicz interpolation:

$$\|Tf\|_{L^p} \leq C_p\,\|(Tf)^\sharp\|_{L^p} \leq C_p\,\|f\|_{L^p}\,, \qquad p_0 < p < \infty\,.$$