Define the Zygmund space $C^1(\mathbb{R}^n)$ by the condition $\|f\|_{C^1} < \infty$ where

$$\|f\|_{C^1} = \sup_k 2^k \|\psi_k(D)f\|_{L^\infty}.$$ 

1. Let $\Phi \in \mathcal{S}(\mathbb{R})$ be as in Lecture 3, so that $\hat{\Phi} = \phi$. Show that

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \sin(2^k x) \Phi(x)$$

belongs to $C^1(\mathbb{R})$ but that there is no finite constant $A$ such that

$$|f(x) - f(0)| \leq A |x|,$$

so in particular $f \notin C^{0,1}(\mathbb{R})$. (Hint: split up the sum depending on $|x|$.)

Proof. To see $f \in C^1(\mathbb{R})$, write $\sin(2^k x) = i \exp(-i2^k x) - i \exp(i2^k x)$, so that

$$\hat{f}(\xi) = i \sum_{k=1}^{\infty} \frac{1}{2^k} e^{-i2^k x} \Phi(\xi) - \frac{1}{2^k} e^{i2^k x} \Phi(\xi)$$

$$= i \sum_{k=1}^{\infty} \frac{1}{2^k} \phi(\xi + 2^k) - \frac{1}{2^k} \phi(\xi - 2^k).$$

Since $\phi$ is supported in the set $|\xi| \leq \frac{3}{2}$, it follows that the $k$-th term is supported where $2^k - \frac{3}{2} \leq |\xi| \leq 2^k + \frac{3}{2}$. The function $\psi_j(\xi)$ is supported where $\frac{1}{2}2^j \leq |\xi| \leq \frac{3}{2}2^j$. For $j \geq 3$ these regions overlap only if $k = j$ or $k = j + 1$, so there are only two terms to consider for $\Psi_j * f$,

$$\Psi_j * f = \sum_{k=j}^{j+1} \frac{1}{2^k} \Psi_j * \left( \sin(2^k \cdot) \Phi \right),$$

and then

$$\|\Psi_j * f\|_{L^\infty} \leq \sum_{k=j}^{j+1} \frac{1}{2^k} \|\Psi_j\|_{L^1} \|\sin(2^k \cdot) \Phi\|_{L^\infty} \leq c 2^{-j}.$$

For the second part, split the sum according to $2^k |x| < 1$ and $2^k |x| \geq 1$. For the second case

$$\left| \sum_{2^k \geq |x|^{-1}} \frac{1}{2^k} \sin(2^k x) \Phi(x) \right| \leq \|\Phi\|_{L^\infty} \sum_{2^k \geq |x|^{-1}} \frac{1}{2^k} \leq 2 \|\Phi\|_{L^\infty} |x|.$$

For the first term, consider $x > 0$ (since $f$ is odd in $x$ the other case follows), and use that for $0 \leq t \leq 1$, $\sin(t) > \frac{1}{2} t$. Also, $\Phi(0) = \int \phi(\xi) \, d\xi > 1$, so
\( \Phi(x) > 1 \) near 0. Thus, for \( x \) small,
\[
\sum_{2^k < |x|^{-1}} \frac{1}{2^k} \sin(2^k x) \Phi(x) \geq \sum_{2^k < |x|^{-1}} x \gtrsim |\log |x|| |x|.
\]
If \( x \) is small this dominates the terms where \( 2^k|x| \geq 1 \), so the combined sum is not less than \( A |x| \) for any finite \( A \).

2. If \( f \in C^1 \), show that for \( |x - y| \leq \frac{1}{2} \)
\[
|f(x) - f(y)| \leq A |\log |x - y|| |x - y|, \quad A \leq C \|f\|_{C^1}.
\]
(Hint: follow the lines of the proof on page 8 of Lecture 3 characterizing \( \|f\|_{C^s} \) for \( 0 < s < 1 \) using the Littlewood-Paley decomposition of \( f \).) The example of Problem 2 shows that this bound cannot be improved.

Proof. Suppose \( f \in C^1 \) and write
\[
f(x) - f(y) = \sum_{k=1}^{\infty} \left( \Psi_k * f \right)(x) - \left( \Psi_k * f \right)(y).
\]
The assumption \( f \in C^1 \) and Bernstein’s lemma give
\[
\|\Psi_k * f\|_{L^\infty} \leq C 2^{-k}\|f\|_{C^1}, \quad \|\nabla(\Psi_k * f)\|_{L^\infty} \leq C \|f\|_{C^1}.
\]
From the first we get that
\[
\left| \sum_{2^k |x - y| \geq 1} \left( \Psi_k * f \right)(x) - \left( \Psi_k * f \right)(y) \right| \leq 4 C \|f\|_{C^1} |x - y|.
\]
From the second
\[
\left| \sum_{2^k |x - y| < 1} \left( \Psi_k * f \right)(x) - \left( \Psi_k * f \right)(y) \right| \leq C \|f\|_{C^1} \sum_{2^k |x - y| < 1} |x - y|
\leq C \|f\|_{C^1} |\log |x - y|| |x - y|.
\]

3. Suppose that \( u \in C^\infty(\overline{\Omega}) \). Let \( u_0(x) = \begin{cases} u(x), & x \in \Omega, \\ 0, & x \notin \Omega. \end{cases} \)

- If \( u(x) = 0 \) for all \( x \in \partial \Omega \), show that \( u_0 \in H^1(\mathbb{R}^n) \), and \( \nabla u = (\nabla u)_0 \).
- If \( u(x) \neq 0 \) for some \( x \in \partial \Omega \), show that \( u_0 \notin H^1(\mathbb{R}^n) \).
- Show that \( u_0 \in H^2(\mathbb{R}^n) \) iff \( u(x) = 0 \) and \( \hat{n} \cdot \nabla u(x) = 0 \) for all \( x \in \partial \Omega \).

Proof. Suppose \( f \in S(\mathbb{R}^n) \).
• The distributional derivative $\partial_j(u_0)$ acting on $f$ is defined as

$$- \int u_0(x) \partial_j f(x) \, dx = - \int_\Omega u(x) \partial_j f(x) \, dx$$

$$= \int_\Omega \partial_j u(x) f(x) \, dx - \int_{\partial\Omega} u(x) f(x) \hat{n}_j(x) \, d\sigma(x)$$

$$= \int (\partial_j u_0)(x) f(x) \, dx$$

where we use that $u = 0$ on $\partial\Omega$ to see the boundary integral is 0.

• Since $(\partial_j u_0) \in L^2$, we show $\partial_j(u_0) - (\partial_j u)_0 \notin L^2$ for some $j$. Choose $j$ so that $\hat{n}_j(x)u(x) \neq 0$ for some $x \in \partial\Omega$. Then by the preceding calculation

$$(\partial_j(u_0) - (\partial_j u)_0)(f) = - \int_{\partial\Omega} u(x) f(x) \hat{n}_j(x) \, d\sigma(x)$$

so it suffices to argue that the last term cannot be bounded by $C\|f\|_{L^2}$ for any $C$. One way is to work in local coordinates in which $\partial\Omega$ is defined by $x_n = 0$, and show that if $g(x') \in C^\infty(\mathbb{R}^{n-1})$ and $g(x_0) \neq 0$, then one can find $f_j \in C^\infty_c(\mathbb{R}^n)$ such that $\lim_{j \to \infty} \|f_j\|_{L^2(\mathbb{R}^n)} = 0$ but $\lim_{j \to \infty} \int g(x')f_j(x',0) \, dx \neq 0$. For this, take $f_j(x) = f(x')\phi(jx_n)$, with $\phi \in C^\infty_c(\mathbb{R})$ satisfying $\phi(0) = 1$, and $f \in C^\infty_c(\mathbb{R}^{n-1})$ satisfying $\int g(x') f(x') \, dx \neq 0$.

• $(\Delta u)_0 \in L^2(\mathbb{R}^n)$, so we show $\Delta(u_0) - (\Delta u)_0 \in L^2(\mathbb{R}^n)$ iff $u(x) = 0$ and $\hat{n} \cdot \nabla u(x) = 0$ for all $x \in \partial\Omega$. Write

$$(\Delta(u_0) - (\Delta u)_0)(f) = \int_\Omega u(x) \Delta f(x) - f(x) \Delta u(x) \, dx$$

$$= \int_{\partial\Omega} u(x) \hat{n}_x(x) \cdot \nabla f(x) - f(x) \hat{n}_x(x) \cdot \nabla u(x) \, d\sigma(x).$$

If $u$ and $\hat{n} \cdot u$ vanish on $\partial\Omega$, then the last term vanishes, so $\Delta(u_0) = (\Delta u)_0 \in L^2(\mathbb{R}^n)$. Consequently $u \in H^2(\mathbb{R}^n)$ by elliptic regularity, since it is compactly supported.

It remains to show that if either $u(x) \neq 0$ or $\hat{n} \cdot \nabla u(x) \neq 0$ for some $x \in \partial\Omega$ then the right hand side is not bounded by $C\|f\|_{L^2}$. This can be done as above, by taking local coordinates where $\partial\Omega = \{x_n = 0\}$ and $\partial x_n = \hat{n}(x)$ at $x_n = 0$. Then choose $\phi$ so either $\phi(0) = 0$ and $\phi'(0) = 1$, or $\phi(0) = 1$ and $\phi'(0) = 0$, depending on whether $u(x_0) \neq 0$ or $\hat{n}(x_0) \cdot \nabla u(x_0) \neq 0$. 