STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS

Throughout, we let $[a,b]$ be a bounded interval in $\mathbb{R}$. $C^2([a,b])$ denotes the space of functions with derivatives of second order continuous up to the endpoints. $C^2_c([a,b])$ is the subspace of functions that vanish near the endpoints.

Let L denote a second order differential operator of the form

$$
\begin{aligned}
Lu(x) &= r(x)u''(x) + r'(x)u'(x) + q(x)u(x) \\
&= \frac{d}{dx}\Big(r(x)\,\frac{du}{dx}\Big) + q(x)u(x)\,.
\end{aligned}
\tag{1}
$$

We assume that $r \in C^1([a,b])$ and $q \in C^0([a,b])$ are real, and that $r(x) \geq c$ for some $c > 0$.

The operator L is the most general second order real ODE which is *formally self-adjoint* on $L^2(dx)$, in that

$$\int_a^b (Lu)\,v\,dx = \int_a^b u\,(Lv)\,dx \qquad \forall \;\; u,v \in C^2_c([a,b])\,.$$

The condition $u,v \in C^2_c([a,b])$ implies that when integrating by parts the boundary terms vanish. Since L has real coefficients, conjugating v or not does not affect the definition.

For general $u,v \in C^2([a,b])$,

$$\int_a^b (Lu)v - u(Lv)\,dx = r\big(u'\,v - u\,v'\big)\big|_a^b \tag{2}$$

and we need to impose first order conditions on u,v at the endpoints to make the right hand side vanish.

A *boundary condition* B is an expression of the form

$$Bu = \alpha u(a) + \beta u(b) + \gamma u'(a) + \delta u'(b)$$

for real constants $\alpha,\beta,\gamma,\delta$. We will impose two conditions $B_1u = 0$ and $B_2u = 0$ where B_1 and B_2 are independent (i.e. the corresponding vectors $(\alpha,\beta,\gamma,\delta)$ are independent), chosen to guarantee that the right hand side of (2) vanish.

Definition 1. The boundary conditions B_1, B_2 are self-adjoint for L if, for all $u,v \in C^2([a,b])$ which satisfy $B_1u = B_2u = B_1v = B_2v = 0$, then

$$\int_a^b (Lu)\,v\,dx = \int_a^b u\,(Lv)\,dx\,.$$

In other words, the vanishing of B_ju and B_jv implies the right-hand side of (2) vanishes.

- *Dirichlet conditions:* $B_1 u = u(a)\,, \quad B_2 u = u(b)\,.$
- *Neumann conditions:* $B_1 u = u'(a)\,, \quad B_2 u = u'(b)\,.$
- *Robin conditions:* $B_1 u = u'(a) - \alpha u(a)\,, \quad B_2 u = u'(b) + \beta u(b)\,, \quad \alpha, \beta > 0.$

The above are *separated boundary conditions*, in that B_1 is a condition at a and B_2 is a condition at b. Any pair of separated conditions is self-adjoint for general L. The most common non-separated condition is

- *Periodic conditions:* $B_1 u = u(b) - u(a)\,, \quad B_2 u = u'(b) - u'(a)\,.$

These are self-adjoint for L if $r(b) = r(a)$.

- Another way to state self-adjointness is to consider the subspace

$$C_B^2([a,b]) = \big\{\, u \in C^2([a,b]) \,:\, B_1 u = B_2 u = 0 \,\big\}\,.$$

Then (L, B_1, B_2) is self-adjoint provided that $\langle Lu, v\rangle = \langle u, Lv\rangle$ for $u, v \in C_B^2([a,b])$, where $\langle u, v\rangle = \int_a^b u\,\bar v\,dx\,.$

We next fix a positive weight function $\rho(x) \in C^2([a,b])$, so $\rho(x) \geq c > 0$ for $x \in [a,b]$, and consider the *Sturm-Liouville eigenvalue problem*

$$Lu = \lambda\,\rho\,u\,, \qquad B_1 u = B_2 u = 0\,.$$

We say that the number λ is an eigenvalue if there is a nonzero solution $u \in C^2([a,b])$ to this equation, and call u an eigenfunction.

Lemma 2. *Let (L, B_1, B_2, ρ) be a self-adjoint Sturm-Liouville system.*

a. *The associated eigenvalues are all real numbers.*
b. *Eigenfunctions associated to different eigenvalues are orthogonal in the inner product*

$$\langle u, v\rangle_\rho = \int_a^b u(x)\bar v(x)\,\rho(x)\,dx\,.$$

c. *The dimension of each eigenspace is at most 2; if the boundary conditions are separated then it is exactly 1.*

Proof. Note that an eigenfunction u is an eigenvector for the operator $\rho^{-1}L$, i.e. $\rho^{-1}Lu = \lambda u$, and that $\langle \rho^{-1}Lu, v\rangle_\rho = \langle u, \rho^{-1}Lv\rangle_\rho$, so that $\rho^{-1}L$ is self-adjoint on the domain $C_B^2([a,b])$ with respect to the inner product $\langle\cdot\,,\cdot\rangle_\rho$. The proof of a and b then follow exactly as for finite dimensional operators.

For c, we note that the space of solutions to $(L - \lambda\rho)u = 0$ is a 2-dimensional subspace of $C^2([a,b])$. If one imposes a separated condition $B_1 u = 0$, this restricts the initial conditions $\big(u(a), u'(a)\big)$ to a 1-dimensional space, hence there is at most a 1-dimensional space of solutions to $(L - \lambda\rho)u = 0$ with the boundary conditions imposed. □

Theorem 3. *Given a self-adjoint Sturm-Liouville system as above, there is an orthonormal basis for the space $L_\rho^2([a,b])$ consisting of eigenfunctions for the Sturm-Liouville problem. The eigenvalues satisfy $\lambda_n \to -\infty$.*

Here, $L^2_\rho([a,b])$ is the space of measurable u on $[a,b]$ such that

$$\|u\|^2_{L^2_\rho} = \int_a^b |u(x)|^2\,\rho(x)\,dx < \infty\,.$$

Since $\rho(x)$ is bounded above and below, this is the same space of functions as $L^2([a,b])$, but the norm and inner product $\big\langle\cdot\,,\cdot\big\rangle_\rho$ are different. The map $u \to \rho^{\frac12} u$ is easily seen to be a unitary map of L^2_ρ onto L^2: $\|\rho^{\frac12} u\|_{L^2} = \|u\|_{L^2_\rho}$. In particular, $\{u_j\}_{j=1}^\infty$ is an orthonormal basis for L^2_ρ iff $\{\rho^{\frac12} u_j\}_{j=1}^\infty$ is an orthonormal basis for L^2 .

We will prove Theorem 3 in the case of separated boundary conditions for simplicity, but it holds for general self-adjoint boundary conditions. We start the proof by reducing to the case where $\rho = 1$. Consider the operator

$$\tilde L u = \rho^{-\frac12} L(\rho^{-\frac12} u) = \frac{d}{dx}\left(\frac{r}{\rho}\,\frac{du}{dx}\right) + \tilde q u\,, \qquad \tilde q = \rho^{-\frac12} L(\rho^{-\frac12})\,,$$

which is formally self-adjoint on $L^2(dx)$, and $Lu = \lambda\rho u$ iff $\tilde L(\rho^{\frac12} u) = \lambda\,\rho^{\frac12} u$. We also define boundary conditions $\tilde B_j(u) = B_j(\rho^{-\frac12} u)$; it follows easily that $\tilde B_1\,,\tilde B_2$ are self-adjoint for $\tilde L$.

We conclude there is orthonormal basis for $L^2(\rho dx)$ of eigenfunctions for $\rho^{-1}L$ satisfying B_j iff there is an orthonormal basis for $L^2(dx)$ consisting of eigenfunctions for $\tilde L$ satisfying $\tilde B_j$, where the bases are related by multiplying by $\rho^{\frac12}$.

We thus assume $\rho = 1$, and consider the eigenfunction problem $Lu = \lambda u$, where $Lu = (ru')' + qu$, and we impose self-adjoint conditions $B_1 u = B_2 u = 0$.

Lemma 4. *If λ is an eigenvalue for $Lu = \lambda u$, then $\lambda \le C$ for some constant C depending on (L, B_1, B_2).*

Proof. Integrating by parts we have

$$\lambda\int_a^b |u|^2\,dx = \int_a^b (Lu)\bar u\,dx = \int_a^b -r|u'|^2 + q\,|u|^2\,dx + r(x)u(x)u'(x)\big|_{x=a}^{x=b}\,.$$

For Dirichlet or Neumann conditions, the boundary terms vanish, and

$$\lambda\int_a^b |u|^2\,dx \le \int_a^b q\,|u|^2\,dx \le \Big(\max_{[a,b]} q\Big)\int_a^b |u|^2\,dx\,. \tag{3}$$

For Robin conditions $u'(a) = \alpha u(a)\,, u'(b) = -\beta u(b)$, we get

$$\lambda\int_a^b |u|^2\,dx = \int_a^b -r|u|^2 + q\,|u|^2\,dx - \alpha r(a)|u(a)|^2 - \beta r(b)|u(b)|^2\,.$$

In the physically realistic case $\alpha, \beta \geq 0$, then (3) still applies. If one or both is negative, we need more work. We bound

$$\max_{[a,b]} u^2 - \min_{[a,b]} u^2 \leq \int_a^b \Big| \frac{d}{dx} u^2 \Big| \, dx = 2 \int_a^b |u'| \, |u| \, dx$$
$$\leq \epsilon \int_a^b |u'|^2 + 2\epsilon^{-1} \int_a^b |u|^2 \, dx \, .$$

Taking ϵ small, for C' sufficiently large we have

$$|\alpha| r(a) |u(a)|^2 + |\beta| r(b) |u(b)|^2 \leq \int_a^b r|u'|^2 + C' \int_a^b |u|^2 \, dx$$

and then

$$\lambda \int_a^b |u|^2 \, dx \leq \int_a^b (q + C') \, |u|^2 \, dx \leq \Big(C' + \max_{[a,b]} q \Big) \int_a^b |u|^2 \, dx \, .$$

□

We replace L by $L - \lambda_0$ with $\lambda_0 = 1 + C$, which has the same eigenfunctions, but with eigenvalues shifted by $-\lambda_0$. We may thus assume all eigenvalues satisfy $\lambda \leq -1$, and in particular

$$Lu = 0 \, , \quad B_1 u = B_2 u = 0 \, , \quad \text{implies} \quad u = 0 \, . \tag{4}$$

We produce eigenfunctions for L by finding eigenfunctions for the operator L^{-1}, which we express as an integral kernel. Thus, we seek to express the solution to

$$Lu = f \, , \quad B_1 u = B_2 u = 0 \, , \quad \text{where} \quad f \in C([a,b]) \, ,$$

in the form

$$u(x) = \int_a^b G(x,y) \, f(y) \, dy \, . \tag{5}$$

The function $G(x,y)$ is called *Green's kernel* for the problem (L, B_1, B_2). We will apply variation of parameters: let u_1 and u_2 be nonzero real solutions to

$$Lu_1 = Lu_2 = 0 \, , \quad B_1 u_1 = 0 \, , \quad B_2 u_2 = 0 \, .$$

Since B_1 and B_2 are separated, then u_1 and u_2 are determined up to a constant multiple. Furthermore, u_1 and u_2 are linearly independent; otherwise they are both solutions to (4). Thus

$$W(x) = \det \begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix} \neq 0 \, .$$

By Abel's theorem, $rW' + r'W = 0$, so $rW = \text{constant}$. The variation of parameters method states that if

$$\begin{bmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{bmatrix} \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ f/r \end{bmatrix}$$

then $u = c_1 u_1 + c_2 u_2$ solves $Lu = f$. This has solution

$$c_1' = \frac{-u_2 f}{rW}\,, \qquad c_2' = \frac{u_1 f}{rW}\,.$$

Since B_1 is a first order operator at a, then $B_1 u = c_1(a) B_1 u_1 + c_2(a) B_1 u_2$, and this vanishes if $c_2(a) = 0$. Similarly $B_2 u = 0$ if $c_1(b) = 0$. Thus we set

$$c_1(x) = \int_x^b \frac{u_2(y)}{rW}\, f(y)\, dy\,, \qquad c_2(x) = \int_a^x \frac{u_1(y)}{rW}\, f(y)\, dy\,.$$

Then (5) holds, where

$$G(x,y) = \begin{cases} (rW)^{-1}\, u_1(x)\, u_2(y)\,, & y \ge x\,, \\ (rW)^{-1}\, u_2(x)\, u_1(y)\,, & x \ge y\,. \end{cases}$$

Note that $G(x,y) = G(y,x)$, and that $G(x,y)$ is continuous on $[a,b] \times [a,b]$. Furthermore, G is real since u_1 and u_2 are. The kernel G is also a left inverse for L, in that if $v \in C^2_B([a,b])$, then

$$v(x) = \int_a^b G(x,y)\, (Lv)(y)\, dy\,.$$

This follows by uniqueness of solutions (4). In particular, if one considers the maps

$$L\,:\, C^2_B([a,b]) \to C([a,b])\,, \qquad G\,:\, C([a,b]) \to C^2_B([a,b])\,,$$

then these maps are respectively the inverse of each other. It follows that G is 1-1 on the space of continuous functions, but we need a stronger result for the diagonalization argument.

Lemma 5. *Suppose that $f \in L^2([a,b])$, and that $\int_a^b G(x,y)\, f(y)\, dy = 0$ for all x. Then $f(y) = 0$ a.e.*

Proof. We will show that $\int f(y)\phi(y)\, dy = 0$ for all $\phi \in C^2_c([a,b])$, and the result follows by density of C^2_c in L^2. We then write

$$\begin{aligned} \int_a^b f(y)\, \phi(y)\, dy &= \int_a^b f(y) \biggl(\int_a^b G(y,x)\, (L\phi)(x)\, dx \biggr) dy \\ &= \int_a^b \biggl(\int_a^b G(x,y)\, f(y)\, dy \biggr) (L\phi)(x)\, dx \end{aligned}$$

using Fubini's theorem. □

The operator $Tf(x) = \int_a^b G(x,y) f(y)\, dy$ is then a self-adjoint, compact operator on $L^2([a,b])$, and 0 is not an eigenvalue of T. There thus exists an orthonormal basis $\{u_j\}_{j=1}^\infty$ for $L^2([a,b])$, where

$$\int_a^b G(x,y)\, u_j(y)\, dy = \nu_j\, u_j(x)\,, \qquad \nu_j \ne 0\,.$$

Since the left hand side is continuous in x so is $u_j(x)$, and thus the left hand side belongs to $C^2_B([a,b])$, and so $u_j \in C^2_B([a,b])$. We then have

$$Lu_j = \lambda_j\, u_j\,, \qquad \lambda_j = \nu_j^{-1}\,,$$

and by Lemma 2 each eigenspace is 1-dimensional (separated boundary conditions). We have arranged that $\lambda_j \leq -1$, which means we can order them so that $\lambda_j \to -\infty$ in a decreasing manner. The eigenvalues for the original problem are $\lambda_j + \lambda_0$, which still decrease to $-\infty$, but there may be finitely many positive eigenvalues, depending on q and the boundary conditions.