1. a. Given a Banach space $B$, and a finite dimensional subspace $V \subset B$, show there exists $x \in B$ such that $||x|| = 1$ and $\text{dist}(x, V) = 1$. (Hint: show there exists $y \in V$ minimizing $\text{dist}(x, y)$, and consider $(x - y)/||x - y||$.)

b. Prove the following: If $T$ is a compact operator on $B$, and $v_j$ a linearly independent sequence of eigenvectors for $T$ with eigenvalues $\lambda_j$, then $\lim_{j \to \infty} \lambda_j = 0$.

2. Consider the operator $Tf(x) = \int_{0}^{1} (x + t) f(t) \, dt$. Find all eigenvalues and eigenvectors of $T$ on $L^2([0, 1])$.

3. Consider the operator $T$ on $C([0, 1])$ defined by $Tf(x) = \int_{0}^{x} f(s) \, ds$.
   a. Show that $T$ is compact, but that $T(B_1)$ is not closed. [Hint: consider the function $\frac{1}{2} - |x - \frac{1}{2}|$.]
   b. Show that the spectral radius of $T$ is 0 by considering $\|T^n\|$.
   c. Characterize 0 as a spectral point: eigenvalue, continuous, or residual.

4. Let $S_R$ and $S_L = S_R^*$ denote the right and left shift operators on $\ell^2(\mathbb{N})$:
   
   $S_R(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, \ldots)$
   $S_L(a_1, a_2, a_3, \ldots) = (a_2, a_3, a_4, \ldots)$

   a. For $S_L$, show that every $z$ with $|z| < 1$ is an eigenvalue of $S_L$, that every $z$ with $|z| = 1$ lies in the continuous spectrum of $S_L$, and that every $z$ with $|z| > 1$ lies in the resolvent set of $S_L$.
   
   b. For $S_R$, show that every $z$ with $|z| < 1$ lies in the residual spectrum of $S_R$, that every $z$ with $|z| = 1$ lies in the continuous spectrum of $S_R$, and that every $z$ with $|z| > 1$ lies in the resolvent set of $S_R$.

5. Let $m(x)$ be a bounded, continuous function on $\mathbb{R}$, and consider the map $T_m$ on $L^2(\mathbb{R})$ where $(T_m f)(x) = m(x) f(x)$.
   
   a. Under what condition on $m$ does $T_m$ have point spectra?
   
   b. Show that the spectrum of $T_m$ consists of the closure of the range of $m(x)$, and that there is no residual spectrum (i.e., the spectrum consists of point and continuous spectra only.)