The sine and cosine series are the orthonormal bases for $L^2([0, \pi], dx)$ given respectively by

$$\left\{ \sqrt{\frac{2}{\pi}} \sin(nx) \right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{ \sqrt{\frac{1}{\pi}} \bigcup \sqrt{\frac{2}{\pi}} \cos(nx) \right\}_{n=1}^{\infty}.$$ 

You may assume that these are orthonormal bases. Let the corresponding expansions of $f \in L^2([0, \pi])$ be given by

$$f(x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} A_n \sin(nx) \quad \text{and} \quad f(x) = \sqrt{\frac{1}{\pi}} B_0 + \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} B_n \cos(nx).$$

Below, $C^k([0, \pi])$ refers to functions in $C^k((0, \pi))$ whose derivatives up to order $k$ agree with continuous functions on the closed interval $[0, \pi]$.

1. a. Suppose that $f \in C^1([0, \pi])$. Show that $\{B_n\} \in \ell^1(\mathbb{N})$. Show, however, that $\{A_n\} \notin \ell^1(\mathbb{N})$ unless $f(0) = f(\pi) = 0$.

b. If $f \in C^2([0, \pi])$, show that $\{nB_n\} \in \ell^1(\mathbb{N})$ if and only if $f'(0) = f'(\pi) = 0$.

2. Let $u(t, x) \in C^2((0, \infty) \times [0, \pi])$ satisfy the heat equation $u_t(t, x) = u_{xx}(t, x)$ on $(0, \infty) \times (0, \pi)$.

a. Show that if $u_x(t, 0) = u_x(t, \pi) = 0$ for all $t > 0$, then for $t > 0$

$$\frac{d}{dt} \int_0^\pi u(t, x) \, dx = 0.$$ 

b. Give an example to show this is not necessarily true if $u$ instead satisfies the boundary conditions $u(t, 0) = u(t, \pi) = 0$.

3. For $f \in L^2([0, \pi])$ let $u(t, x)$ for $(t, x) \in (0, \infty) \times \mathbb{R}$ be given by

$$u(t, x) = \sqrt{\frac{1}{\pi}} B_0 + \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} B_n \cos(nx) e^{-n^2t}.$$ 

a. Show that $\lim_{t \to 0} ||u(t, x) - f(x)||_{L^2([0, \pi], dx)} = 0$.

b. Show that $\lim_{t \to \infty} ||u(t, x) - c||_u = 0$, where $c = \frac{1}{\pi} \int_0^\pi f(x) \, dx$, and $|| \cdot ||_u$ is the uniform norm with respect to $x$ on $C([0, \pi])$. 