Distributions of Compact Support

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Division in 1-d

Lemma

If $u \in \mathcal{D}'(\mathbb{R})$ and $x^m u = 0$, then $u = \sum_{j=0}^{m-1} c_j \, \partial^j \delta$.

Proof. Fix $h(x) \in C_c^\infty(\mathbb{R})$ s.t. $h = 1$ on an interval containing 0. Then $\langle u, \phi \rangle = \langle u, h\phi \rangle$, since one can write $\phi - h\phi = x^m \tilde{\psi}$.

Next write $\quad h(x)\phi(x) = \sum_{j=0}^{m-1} \frac{1}{j!} \, \partial^j \phi(0) x^j h(x) \, + \, x^m h(x) \psi(x)$

Since $\langle u, x^m h\psi \rangle = 0$, then

$$\langle u, \phi \rangle = \sum_{j=0}^{m-1} \frac{1}{j!} \, \partial^j \phi(0) \langle u, x^j h \rangle = \sum_{j=0}^{m-1} c_j \, \langle \partial^j \delta, \phi \rangle$$

where $c_j = (-1)^j \langle u, x^j h \rangle / j!$ □

Division in 1-d

Theorem

Define $u_m = \frac{(-1)^{m-1}}{(m-1)!}\partial^m \ln|x|$. Then $x^m u_m = 1\,,\quad m \geq 1\,.$

Proof. Write $u_m = (1-m)^{-1}\partial u_{m-1}$. Result true for $u_1 = \text{p.v.}\frac{1}{x}$, so assume $x^{m-1}u_{m-1} = 1$.

$$\begin{aligned} x^m\,\partial u_{m-1} &= \partial(x^m u_{m-1}) - m\,x^{m-1}\,u_{m-1} \\ &= \partial(x) - m \\ &= 1 - m \qquad \square \end{aligned}$$

Remark. By Lemma, the solution to $x^m u = 1$ is determined up to derivatives of δ of order $\leq m-1$, so classified all solutions.

Division by polynomials on $\mathbb{R}$

Theorem

If $p(x)$ is a polynomial, $\exists\, u \in \mathcal{D}'(\mathbb{R})$ such that $p(x)u = 1$.

Proof. Use principal value decomposition of $1/p(z)$, $z \in \mathbb{C}$,

$$\frac{1}{p(z)} = \sum_{p(z_j)=0} \frac{q_j(z)}{(z - z_j)^{m_j}}$$

and take $u_j \in \mathcal{D}'(\mathbb{R}) \; : \; (z - z_j)^{m_j} u_j = 1$ as distributions.

Then $p(z)u_j = p_j(z)$, where $p(z) = p_j(z)\,(z - z_j)^{m_j}$, so

$$p(z) \sum_{p(z_j)=0} q_j(z)u_j = \sum_{p(z_j)=0} q_j(z)p_j(z) = 1\,.$$

Compactly supported distributions

Definition

$\mathcal{E}'(\Omega)$ is the subspace of $f \in \mathcal{D}'(\Omega)$ such that $\mathrm{supp}(f) \Subset \Omega$.

Suppose that supp(f) is compact in Ω, and take $\chi \in C_c^\infty(\Omega)$ such that $\chi(x) = 1$ on an open neighborhood of $\mathrm{supp}(f)$. Then for all $\phi \in C_c^\infty(\Omega)$,

$$\langle f\,, \phi\rangle = \langle f\,, \chi\phi\rangle$$

since $\mathrm{supp}\big((1-\chi)\phi\big) \cap \mathrm{supp}(f) = \emptyset$.

$\langle f\,, \chi\phi\rangle$ *makes sense if* $\phi \in C^\infty(\Omega)$, *since then* $\chi\phi \in C_c^\infty(\Omega)$.

Topology on $C^\infty(\Omega)$.

- If $K \Subset \Omega$, define: $$\|\phi\|_{K,m} = \sup_{x\in K} \sup_{|\alpha|\le m} \left|\partial_x^\alpha \phi(x)\right| .$$

- If $K_1 \subset K_2 \subset K_3 \subset \cdots$, and $\bigcup_j \operatorname{int}(K_j) = \Omega$, then the countable family of seminorms $\|\cdot\|_{K_j,m}$ makes $C^\infty(\Omega)$ into a Frechét space, and $\phi_n \to \phi$ in this topology iff

$$\lim_{n\to\infty} \sup_{x\in K} \left|\partial_x^\alpha \phi_n(x) - \partial_x^\alpha \phi(x)\right| = 0$$

for each α, K.

- *Topology of uniform convergence on compact sets.*

Pairing of $\mathcal{E}'(\Omega)$ and $C^\infty(\Omega)$.

Theorem

The continuous linear maps from $C^\infty(\Omega) \to \mathbb{C}$ are 1-1 identified with $\mathcal{E}'(\Omega)$, where $f \in \mathcal{E}'(\Omega)$ acts on $\phi \in C^\infty(\Omega)$ by

$$\langle f , \phi \rangle \;=\; \langle f , \chi\phi \rangle\,, \quad \chi \in C_c^\infty(\Omega) \text{ equals 1 on nbhd of } \operatorname{supp}(f)\,.$$

Proof. $f \in \mathcal{E}'(\Omega)$, map $\phi \to \langle f , \chi\phi \rangle$ same for each choice of χ, since all such $\chi\phi$ agree on nbhd of $\operatorname{supp}(f)$. If $K = \operatorname{supp}(\chi)$,

$$\begin{aligned} \big|\langle f , \chi\phi \rangle\big| \;&\le\; C_K \sup_{|\alpha|\le m_K} \sup_x \big|\partial_x^\alpha(\chi\phi)(x)\big| \\ &\le\; C_\chi \sup_{|\alpha|\le m_K} \sup_{x\in K} \big|\partial_x^\alpha \phi(x)\big| \end{aligned}$$

Map $\phi \to \langle f , \chi\phi \rangle$ continuous in seminorm topology on $C^\infty(\Omega)$.

Pairing of $\mathcal{E}'(\Omega)$ and $C^\infty(\Omega)$.

Conversely, suppose $f : C^\infty(\Omega) \to \mathbb{C}$ is continuous, so $\exists\, K\,,\, m$:

$$(*) \qquad \big|\langle f\,,\phi\rangle\big| \;\le\; C\,\|\phi\|_{K,m}$$

If $\chi \in C_c^\infty(\Omega)$ equals 1 on nbhd of K , then $\|\phi - \chi\phi\|_{K,m} = 0$, so

$$\langle f\,,\phi\rangle = \langle f\,,\chi\phi\rangle$$

By $(*)$, the restriction of f to $C_c^\infty(\Omega)$ is in $\mathcal{D}'(\Omega)$, and necessarily is supported in K. □