The Resolvent of an Operator

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Notation

- X, Y are Banach spaces over $\mathbb{C}$: normed vector spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, complete as metric spaces.
- $\mathcal{B}(X, Y)$ is space of bounded linear operators from X to Y with norm
$$\|T\|_{\mathcal{B}(X,Y)} = \sup_{x\neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$
$\|T\|$ is smallest constant C such that $\|Tx\|_Y \leq C\,\|x\|_X$.
- $\mathcal{B}(X, Y)$ is itself a Banach space. Completeness in the norm depends on completeness of Y.
- The dual X^* of X is the space of linear maps $f : X \to \mathbb{C}$, which is a Banach space under the norm
$$\|f\|_{X^*} = \sup_{x\neq 0} \frac{|f(x)|}{\|x\|_X}$$

Power series in Banach spaces : take $z \in \mathbb{C}$

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n a_n \,, \qquad a_n \in X\,.$$

Theorem

Let $R^{-1} = \limsup_{n\to\infty} \|a_n\|^{\frac{1}{n}}$. Then $f(z)$ converges if $|z - z_0| < R$, and defines a continuous function $f : \{z : |z - z_0| < R\} \to X$.

Proof. Let $0 < r_0 < r < R$, so $\|a_n\| \le r^{-n}$ if n sufficiently large.

- If $|z - z_0| \le r_0 \;\Rightarrow\; \|(z - z_0)^n a_n\| \le \left(\frac{r_0}{r}\right)^n$, so $f(z)$ converges.
- If $f_N(z) = \sum_{n=0}^{N} (z - z_0)^n a_n$, then

$$\sup_{|z-z_0|\le r_0} \|f(z) - f_N(z)\| \le \sum_{n=N+1}^{\infty} \left(\frac{r_0}{r}\right)^n \to 0 \quad \text{as} \quad N \to \infty$$

so $f(z)$ is uniform limit of continuous functions on $|z - z_0| \le r_0$.

Holomorphic functions $f(z) : \Omega \to X$, $\Omega \subset \mathbb{C}$ open.

Definition

$f(z) : \Omega \to X$ is holomorphic if $f'(z) = \lim_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$

exists for each $z \in \Omega$, and $f'(z)$ is continuous from $\Omega \to X$.

Example: $f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \, a_n \,, \qquad \Omega = \{ z : |z - z_0| < R \}$

Cauchy integral formula: $f(z)$ holomorphic on $|z - z_0| < R$

$$f(z) \;=\; \frac{1}{2\pi i} \oint_{|w - z_0| = r} \frac{f(w)}{w - z} \, dw \qquad \text{if} \quad |z - z_0| < r < R$$

Corollary to CIF: If $f(z)$ holomorphic on $|z - z_0| < R$, then

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \, a_n \,, \qquad |z - z_0| < R \,,$$

where $\quad a_n = \dfrac{1}{2\pi i} \oint_{|w - z_0| = r} \dfrac{f(w)}{(w - z_0)^{n+1}} \, dw \quad$ any $r < R$

and $\qquad \limsup_{n \to \infty} \|a_n\|^{\frac{1}{n}} \leq R^{-1}$.

Conclude: $f(z)$ holomorphic on $|z - z_0| < R$, and $\exists\, r > 0$:

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n \, a_n \quad \text{for } |z - z_0| < r$$

then $\quad \limsup_{n \to \infty} \|a_n\|^{\frac{1}{n}} \leq R^{-1}$.

Invertible operators in $\mathcal{B}(X)$

- We say $S \in \mathcal{B}(X)$ is **invertible** if $\exists\, S^{-1} \in \mathcal{B}(X)$ such that

$$S^{-1} S = S\, S^{-1} = \mathrm{I}$$

Theorem

The invertible operators form an open subset of $\mathcal{B}(X)$.

Proof. If T_0 is invertible, and $\|S\| < \|T_0^{-1}\|^{-1}$, formally write

$$(T_0 - S)^{-1} = (\mathrm{I} - T_0^{-1} S)^{-1} T_0^{-1} = \sum_{n=0}^{\infty} (T_0^{-1} S)^n T_0^{-1}$$

Converges in $\mathcal{B}(X)$ since $\|T_0^{-1} S\| < 1$. Is left inverse since

$$\sum_{n=0}^{\infty} (T_0^{-1} S)^n T_0^{-1} (T_0 - S) = \sum_{n=0}^{\infty} (T_0^{-1} S)^{-n} - \sum_{n=1}^{\infty} (T_0^{-1} S)^{-n} = \mathrm{I}$$

Can similarly verify is right inverse.

Resolvent set of $T \in \mathcal{B}(X)$

Definition

If $T \in \mathcal{B}(X)$, the resolvent set $\rho(T) \subset \mathbb{C}$ is set of z such that $(z\mathrm{I} - T)$ is invertible. Let $R_T(z) = (z\mathrm{I} - T)^{-1}$ for $z \in \rho(T)$.

- If $z_0 \in \rho(T)$ and $|z - z_0| < \|(z_0\mathrm{I} - T)^{-1}\|^{-1}$, then

$$
\begin{aligned}
\big(z\mathrm{I} - T\big)^{-1} &= \big((z_0\mathrm{I} - T) + (z - z_0)\,\mathrm{I}\big)^{-1} \\
&= \sum_{n=0}^{\infty} (-1)^n (z - z_0)^n (z_0\mathrm{I} - T)^{-(n+1)}
\end{aligned}
$$

- Shows that: $R_T(z)$ is holomorphic map from $\rho(T) \to \mathcal{B}(X)$.

Theorem

Let $r_\sigma(T) = \limsup_{n\to\infty} \left\| T^n \right\|^{\frac{1}{n}}$. Then $\rho(T) \supseteq \{z : |z| > r_\sigma(T)\}$.

Proof. We'll show that $\left(w^{-1}\mathrm{I} - T\right)^{-1}$ exists for $|w| < r_\sigma(T)^{-1}$

Formally:
$$\begin{aligned}\left(w^{-1}\mathrm{I} - T\right)^{-1} &= w\left(\mathrm{I} - wT\right)^{-1} \\ &= w\sum_{n=0}^{\infty} w^n T^n\end{aligned}$$

which converges in $\mathcal{B}(X)$ if $|w| < \left(\limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}}\right)^{-1}$.

It gives a true inverse where it converges, since then:

$$\left(w^{-1}\mathrm{I} - T\right) w \sum_{n=0}^{\infty} w^n T^n = \sum_{n=0}^{\infty} w^n T^n - \sum_{n=1}^{\infty} w^n T^n = \mathrm{I}\,.$$

Spectral radius

Remark: Formula for radius of convergence shows that $r_\sigma(T)$ is smallest value of r such that $(I - wT)^{-1}$ exists for all $|w| < r^{-1}$, thus is smallest value so $(zI - T)^{-1}$ exists for all $|z| > r$.

Definition

The *spectrum* of $T \in \mathcal{B}(X)$ is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.

- $\sigma(T)$ is a closed subset of $\mathbb{C}$.
- $\sigma(T) \subset \{z : |z| \le r_\sigma(T)\}$
- $\exists\, z \in \sigma(T)$ with $|z| = r_\sigma(T) = \limsup_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.