Linear ODE

Let $I \subset \mathbb{R}$ be an interval (open or closed, finite or infinite — at either end). Suppose $A : I \rightarrow \mathbb{R}^{n \times n}$ and $b : I \rightarrow \mathbb{R}^n$ are continuous. The DE

$$(*) \quad x' = A(t)x + b(t)$$

is called a first-order linear [system of] ODE[s] on $I$. Since $f(t, x) \equiv A(t)x + b(t)$ is continuous in $t, x$ on $I \times \mathbb{R}^n$ and, for any compact subinterval $[c, d] \subset I$, $f$ is uniformly Lipschitz in $x$ on $[c, d] \times \mathbb{R}^n$ (with Lipschitz constant $\max_{c \leq t \leq d} |A(t)|$), we have global existence and uniqueness of solutions of the IVP

$$x' = A(t)x + b(t), \quad x(t_0) = x_0$$

on all of $I$ (where $t_0 \in I$, $x_0 \in \mathbb{R}^n$).

If $b \equiv 0$ on $I$, $(*)$ is called a linear homogeneous system (LH).

If $b \not\equiv 0$ on $I$, $(*)$ is called a linear inhomogeneous system (LI).

**Fundamental Theorem for LH.** The set of all solutions of (LH) $x' = A(t)x$ on $I$ forms an $n$-dimensional vector space over $\mathbb{R}$ (in fact, a subspace of $C^1(I, \mathbb{R}^n)$).

**Proof.** Clearly $x'_1 = Ax_1$ and $x'_2 = Ax_2$ imply $(c_1x_1 + c_2x_2)' = A(c_1x_1 + c_2x_2)$, so the set of solutions of (LH) forms a vector space over $\mathbb{R}$, which is clearly a subspace of $C^1(I, \mathbb{R}^n)$. Fix $\tau \in I$, and let $y_1, \ldots, y_n$ be a basis for $\mathbb{R}^n$. For $1 \leq j \leq n$, let $x_j(t)$ be the solution of the IVP $x' = Ax$, $x(\tau) = y_j$. Then $x_1(t), \ldots, x_n(t)$ are linearly independent in $C^1(I, \mathbb{R}^n)$; indeed,

$$\sum_{j=1}^n c_jx_j(t) = 0 \quad \text{in} \quad C^1(I, \mathbb{R}^n)$$

$$\Rightarrow$$

$$\sum_{j=1}^n c_jx_j(t) = 0 \quad \forall \quad t \in I$$

$$\Rightarrow$$

$$\sum_{j=1}^n c_jy_j = \sum_{j=1}^n c_jx_j(\tau) = 0$$

$$\Rightarrow$$

$$c_j = 0 \quad j = 1, 2, \ldots, n.$$

Now if $x(t)$ is any solution of (LH), there exist unique $c_1, \ldots, c_n$ such that $x(\tau) = c_1y_1 + \cdots + c_ny_n$. Clearly $c_1x_1(t) + \cdots + c_nx_n(t)$ is a solution of the IVP

$$x' = A(t)x, \quad x(\tau) = c_1y_1 + \cdots + c_ny_n,$$

so by uniqueness, $x(t) = c_1x_1(t) + \cdots + c_nx_n(t)$ for all $t \in I$. Thus $x_1(t), \ldots, x_n(t)$ span the vector space of all solutions of (LH) on $I$. So they form a basis, and the dimension is $n$. □
Proof. Let \( L : C^1(I, \mathbb{R}^n) \to C^0(I, \mathbb{R}^n) \) by \( Lx = (\frac{d}{dt} - A(t))x \), i.e., \( (Lx)(t) = x'(t) - A(t)x(t) \) for \( x(t) \in C^1(I, \mathbb{R}^n) \). L is called a linear differential operator. The solution space in the previous theorem is precisely the null space of L. Thus the null space of L is finite dimensional and has dimension n.

Remark. Define the linear operator \( L : C^1(I, \mathbb{R}^n) \to C^0(I, \mathbb{R}^n) \) by \( Lx = (\frac{d}{dt} - A(t))x \), i.e., \( (Lx)(t) = x'(t) - A(t)x(t) \) for \( x(t) \in C^1(I, \mathbb{R}^n) \). L is called a linear differential operator. The solution space in the previous theorem is precisely the null space of L. Thus the null space of L is finite dimensional and has dimension n.

Definition. A set \( \{ \varphi_1, \ldots, \varphi_n \} \) of solutions of (LH) \( x' = Ax \) on I is said to be a fundamental set of solutions if it is a basis for the vector space of all solutions. If \( \Phi : I \to \mathbb{R}^{n \times n} \) is an \( n \times n \) matrix function of \( t \in I \) whose columns form a fundamental set of solutions of (LH), then \( \Phi(t) \) is called a fundamental matrix for (LH) \( x' = A(t)x \). Checking columnwise shows that a fundamental matrix satisfies
\[
\Phi'(t) = A(t)\Phi(t).
\]

Definition. If \( X : I \to \mathbb{R}^{n \times k} \) is in \( C^1(I, \mathbb{R}^{n \times k}) \), we say that \( X \) is an \([n \times k]\) matrix solution of (LH) if \( X'(t) = A(t)X(t) \). Clearly \( X(t) \) is a matrix solution of (LH) if and only if each column of \( X(t) \) is a solution of (LH). (We will mostly be interested in the case \( k = n \).)

Theorem. Let \( A : I \to \mathbb{R}^{n \times n} \) be continuous, where \( I \subset \mathbb{R} \) is an interval, and suppose \( X : I \to \mathbb{R}^{n \times n} \) is an \( n \times n \) matrix solution of (LH) \( x' = A(t)x \) on I, i.e., \( X'(t) = A(t)X(t) \) on I. Then \( \det(X(t)) \) satisfies the linear homogeneous first-order scalar ODE
\[
\det(X(t))' = \text{tr}(A(t)) \det(X(t)),
\]
and so for all \( \tau, t \in I \),
\[
\det(X(t)) = (\det(X(\tau))) \exp \int_\tau^t \text{tr}(A(s))ds.
\]

Proof. Let \( x_{ij}(t) \) denote the \( ij \)th element of \( X(t) \), and let \( \tilde{X}_{ij}(t) \) denote the \((n-1) \times (n-1)\) matrix obtained from \( X(t) \) by deleting its \( i \)th row and \( j \)th column. The co-factor representation of the determinant gives
\[
\det(X) = \sum_{j=1}^n (-1)^{(i+j)}x_{ij} \det(\tilde{X}_{ij}), \quad i = 1, 2, \ldots, n.
\]
Hence
\[
\frac{\partial}{\partial x_{ij}} \det(X) = (-1)^{(i+j)} \det(\tilde{X}_{ij}),
\]
and so by the chain rule
\[
(\det(X(t))') = \sum_{j=1}^n (-1)^{(i+j)}x'_{ij}(t) \det(\tilde{X}_{ij}(t)) + \cdots + \sum_{j=1}^n (-1)^{(n+j)}x'_{n j}(t) \det(\tilde{X}_{ij}(t))
\]
\[
= \det \begin{bmatrix} x'_{11} & x'_{12} & \cdots & x'_{1n} \\ \text{(remaining \hspace{1em} x_{ij})} \end{bmatrix} + \cdots + \det \begin{bmatrix} (\text{remaining \hspace{1em} x_{ij}}) \\ x'_{n1} & x'_{n2} & \cdots & x'_{nm} \end{bmatrix}.
\]
Now by (LH)

\[
\begin{bmatrix}
x_1' & x_2' & \cdots & x_{1n}'
\end{bmatrix}
= \left[ \Sigma_k a_{1k} x_{k1} \cdots \Sigma_k a_{1k} x_{kn} \right]
= a_{11} [x_{11} \cdots x_{1n}] + a_{12} [x_{21} \cdots x_{2n}] + \cdots + a_{1n} [x_{n1} \cdots x_{nn}].
\]

Subtracting \(a_{12} [x_{21} \cdots x_{2n}] + \cdots + a_{1n} [x_{n1} \cdots x_{nn}]\) from the first row of the matrix in the first determinant on the RHS doesn’t change that determinant. A similar argument applied to the other determinants gives

\[
(\text{det } X(t))' = \text{det} \begin{bmatrix} a_{11} [x_{11} \cdots x_{1n}] \\
(\text{remaining } x_{ij})
\end{bmatrix} + \cdots + \text{det} \begin{bmatrix} (\text{remaining } x_{ij}) \\
 a_{nn} [x_{n1} \cdots x_{nn}]
\end{bmatrix}
= (a_{11} + \cdots + a_{nn}) \text{ det } X(t) = \text{tr } (A(t)) \text{ det } X(t).
\]

\[\square\]

**Corollary.** Let \(X(t)\) be an \(n \times n\) matrix solution of (LH) \(x' = A(t)x\). Then either

\[
(\forall t \in I) \quad \text{det } X(t) \neq 0 \quad \text{or} \quad (\forall t \in I) \quad \text{det } X(t) = 0.
\]

**Corollary.** Let \(X(t)\) be an \(n \times n\) matrix solution of (LH) \(x' = A(t)x\). Then the following statements are equivalent.

1. \(X(t)\) is a fundamental matrix for (LH) on \(I\).
2. \((\exists \tau \in I) \ \text{det } X(\tau) \neq 0\) (i.e., columns of \(X\) are linearly independent at \(\tau\))
3. \((\forall t \in I) \ \text{det } X(t) \neq 0\) (i.e., columns of \(X\) are linearly independent at every \(t \in I\)).

**Definition.** If \(X(t)\) is an \(n \times n\) matrix solution of (LH) \(x' = A(t)x\), then \(\text{det } (X(t))\) is often called the Wronskian [of the columns of \(X(t)\)].

**Remark.** This is not quite standard notation for general LH systems \(x' = A(t)x\). It is used most commonly when \(x' = A(t)x\) is the first-order system equivalent to a scalar \(n^{th}\)-order linear homogeneous ODE.

**Theorem.** Suppose \(\Phi(t)\) is a fundamental matrix for (LH) \(x' = A(t)x\) on \(I\).

(a) If \(c \in \mathbb{F}^n\), then \(x(t) = \Phi(t)c\) is a solution of (LH) on \(I\).

(b) If \(x(t) \in C^1(I, \mathbb{F}^n)\) is any solution of (LH) on \(I\), then there exists a unique \(c \in \mathbb{F}^n\) for which \(x(t) = \Phi(t)c\).
Proof. The theorem just restates that the columns of $\Phi(t)$ for a basis for the set of solutions of (LH).

Remark. The general solution of (LH) is $\Phi(t)c$ for arbitrary $c \in \mathbb{F}^n$, where $\Phi(t)$ is a fundamental matrix.

Theorem. Suppose $\Phi(t)$ is a fundamental matrix (F.M.) for (LH) $x' = A(t)x$ on $I$.

(a) If $C \in \mathbb{F}^{n \times n}$ is invertible, then $X(t) = \Phi(t)C$ is also a F.M. for (LH) on $I$.

(b) If $X(t) \in C^1(I, \mathbb{F}^{n \times n})$ is any F.M. for (LH) on $I$, then there exists a unique invertible $C \in \mathbb{F}^{n \times n}$ for which $X(t) = \Phi(t)C$.

Proof. For (a), observe that

$$X'(t) = \Phi'(t)C = A(t)\Phi(t)C = A(t)X(t),$$

so $X(t)$ is a matrix solution, and $\det X(t) = (\det \Phi(t))(\det C) \neq 0$. For (b), set $\Psi(t) = \Phi(t)^{-1}X(t)$. Then $X = \Phi \Psi$, so

$$\Phi'\Psi + \Phi\Psi' = (\Phi \Psi)' = X' = AX = A\Phi \Psi = \Phi'\Psi,$$

which implies that $\Phi \Psi' = 0$. Since $\Phi(t)$ is invertible for all $t \in I$, $\Psi'(t) \equiv 0$ on $I$. So $\Psi(t)$ is a constant invertible matrix $C$. Since $C = \Psi = \Phi^{-1}X$, we have $X(t) = \Phi(t)C$.

Remark. If $B(t) \in C^1(I, \mathbb{F}^{n \times n})$ is invertible for each $t \in I$, then

$$\frac{d}{dt} (B^{-1}(t)) = -B^{-1}(t)B'(t)B^{-1}(t).$$

The proof is to differentiate $I = BB^{-1}$:

$$0 = \frac{d}{dt} (I) = \frac{d}{dt} (B(t)B^{-1}(t)) = B(t)\frac{d}{dt} (B^{-1}(t)) + B'(t)B^{-1}(t).$$

Adjoint Systems

Let $\Phi(t)$ be a F.M. for (LH) $x' = A(t)x$. Then

$$(\Phi^{-1})' = -\Phi^{-1}\Phi'\Phi^{-1} = -\Phi^{-1}A\Phi\Phi^{-1} = -\Phi^{-1}A.$$

Taking conjugate transposes, $(\Phi^*)^{-1} = -A^*\Phi^*-1$. So $\Phi^*-1(t)$ is a F.M. for the adjoint system (LH*) $x' = -A^*(t)x$.

Theorem. If $\Phi(t)$ is a F.M. for (LH) $x' = A(t)x$ and $\Psi(t) \in C^1(I, \mathbb{F}^{n \times n})$, then $\Psi(t)$ is a F.M. for (LH*) $x' = -A^*(t)x$ if and only if $\Psi^*(t)\Phi(t) = C$, where $C$ is a constant invertible matrix.

Proof. Suppose $\Psi(t)$ is a F.M. for (LH*). Since $\Phi^*-1(t)$ is also a F.M. for (LH*), $\exists$ an invertible $C \in \mathbb{F}^{n \times n} \ni \Psi(t) = \Phi^*-1(t)C^*$, i.e., $\Psi^* = C\Phi^*-1$, $\Psi^*\Phi = C$. Conversely, if $\Psi^*(t)\Phi(t) = C$ (invertible), then $\Psi^* = C\Phi^*-1$, $\Psi = \Phi^*-1C^*$, so $\Psi$ is a F.M. for (LH*).
Normalized Fundamental Matrices

**Definition.** A F.M. $\Phi(t)$ for (LH) $x' = A(t)x$ is called *normalized at time* $\tau$ if $\Phi(\tau) = I$, the identity matrix. (Convention: if not stated otherwise, a normalized F.M. usually means normalized at time $\tau = 0$.)

**Facts:**

1. For a given $\tau$, the F.M. of (LH) normalized at $\tau$ exists and is unique. (*Proof.* The $j^{th}$ column of $\Phi(t)$ is the solution of the IVP $x' = A(t)x$, $x(\tau) = e_j$.)

2. If $\Phi(t)$ is the F.M. for (LH) normalized at $\tau$, then the solution of the IVP $x' = A(t)x$, $x(\tau) = y$ is $x(t) = \Phi(t)y$. (*Proof.* $x(t) = \Phi(t)y$ satisfies (LH) $x' = A(t)x$, and $x(\tau) = \Phi(\tau)y = Iy = y$.)

3. For any fixed $\tau, t$, the solution operator $S_t^\tau$ for (LH), mapping $x(\tau)$ into $x(t)$, is a *linear* operator on $\mathbb{F}^n$, and its matrix is the F.M. $\Phi(t)$ for (LH) normalized at $\tau$, evaluated at $t$.

4. If $\Phi(t)$ is *any* F.M. for (LH), then for fixed $\tau$, $\Phi(t)\Phi^{-1}(\tau)$ is the F.M. for (LH) normalized at $\tau$. (*Proof.* It is a F.M. taking the value $I$ at $\tau$.) Thus:
   
   (a) $\Phi(t)\Phi^{-1}(\tau)$ is the matrix of the solution operator $S_t^\tau$ for (LH); and
   
   (b) the solution of the IVP $x' = A(t)x$, $x(\tau) = y$ is $x(t) = \Phi(t)\Phi^{-1}(\tau)y$.

**Reduction of Order for (LH) $x' = A(t)x$**

If $m (< n)$ linearly independent solutions of the $n \times n$ linear homogeneous system $x' = A(t)x$ are known, then one can derive an $(n - m) \times (n - m)$ system for obtaining $n - m$ more linearly independent solutions. See Coddington & Levinson for details.

**Inhomogeneous Linear Systems**

We now want to express the solution of the IVP

$$x' = A(t)x + b(t), \quad x(t_0) = y$$

for the linear inhomogeneous system

**(LI)**

$$x' = A(t)x + b(t)$$

in terms of a F.M. for the associated homogeneous system

**(LH)**

$$x' = A(t)x.$$
Variation of Parameters

Let $\Phi(t)$ be any F.M. for (LH). Then for any constant vector $c \in \mathbb{F}^n$, $\Phi(t)c$ is a solution of (LH). We will look for a solution of (LI) of the form

$$x(t) = \Phi(t)c(t)$$

(varying the “constants” — elements of $c$). Plugging into (LI), we want

$$(\Phi c)' = A\Phi c + b,$$

or equivalently

$$\Phi'c + \Phi c' = A\Phi c + b.$$ 

Since $\Phi' = A\Phi$, this gives $\Phi c' = b$, or $c' = \Phi^{-1}b$. So let

$$c(t) = c_0 + \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds$$

for some constant vector $c_0 \in \mathbb{F}^n$, and let $x(t) = \Phi(t)c(t)$. These calculations show that $x(t)$ is a solution of (LI). To satisfy the initial condition $x(t_0) = y$, we take $c_0 = \Phi^{-1}(t_0)y$, and obtain

$$x(t) = \Phi(t)\Phi^{-1}(t_0)y + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s)b(s)ds.$$ 

In words, this equation states that

$$\left\{\begin{array}{l}
\text{soln of (LI)} \\
\text{with I.C. } x(t_0) = y
\end{array}\right\} = \left\{\begin{array}{l}
\text{soln of (LH)} \\
\text{with I.C. } x(t_0) = y
\end{array}\right\} + \left\{\begin{array}{l}
\text{soln of (LI)} \\
\text{with homog. I.C. } x(t_0) = 0
\end{array}\right\}.$$ 

Viewing $y$ as arbitrary, we find that the general solution of (LI) equals the general solution of (LH) plus a particular solution of (LI).

Recall that $\Phi(t)\Phi^{-1}(t_0)$ is the matrix of $S^t_{t_0}$, and $\Phi(t)\Phi^{-1}(s)$ is the matrix of $S^s_{t_0}$. So the above formula for the solution of the IVP can be written just in terms of the solution operator:

**Duhamel’s Principle.** If $S^t_{t_0}$ is the solution operator for (LH), then the solution of the IVP $x' = A(t)x + b(t)$, $x(t_0) = y$ is

$$x(t) = S^t_{t_0}y + \int_{t_0}^{t} S^t_{s}(b(s))ds.$$ 

**Remark.** So the effect of the inhomogeneous term $b(t)$ in (LI) is the same as adding an additional IC $b(s)$ at each time $s \in [t_0, t]$ and integrating these solutions $S^t_{s}(b(s))$ of (LH) with respect to $s \in [t_0, t]$.
**Constant Coefficient Systems**

Consider the linear homogeneous constant-coefficient first-order system

\[(LHC) \quad x' = Ax,\]

where \(A \in \mathbb{F}^{n \times n}\) is a constant matrix. The F.M. of \((LHC)\), normalized at 0, is \(\Phi(t) = e^{tA}\). This is justified as follows. Recall that

\[e^{B} = \sum_{j=0}^{\infty} \frac{1}{j!} B^{j}\]

where \(B^{0} \equiv I\). So \(\Phi(0) = I\). Term by term differentiation is justified in the series for \(e^{tA}\):

\[\Phi'(t) = \frac{d}{dt}(e^{tA}) = \sum_{j=0}^{\infty} \frac{1}{j!} \frac{d}{dt}(tA)^{j}\]

\[= \sum_{j=1}^{\infty} \frac{1}{(j-1)!} t^{j-1} A^{j} = A \sum_{k=0}^{\infty} \frac{1}{k!} (tA)^{k} = A e^{tA} = A \Phi(t).\]

We can express \(e^{tA}\) using the Jordan form of \(A\): if \(P^{-1}AP = J\) is in Jordan form where \(P \in \mathbb{F}^{n \times n}\) is invertible (assume \(\mathbb{F} = \mathbb{C}\) if \(A\) has any nonreal eigenvalues), then \(A = PJP^{-1}\), so \(e^{tA} = e^{tPJP^{-1}} = Pe^{tJ}P^{-1}\). If

\[J = \begin{bmatrix} J_{1} & & 0 \\ & \ddots & \\ 0 & & J_{s} \end{bmatrix}\]

where each \(J_{k}\) is a single Jordan block, then

\[e^{tJ} = \begin{bmatrix} e^{tJ_{1}} & & 0 \\ & \ddots & \\ 0 & & e^{tJ_{s}} \end{bmatrix}.\]

Finally, if

\[J_{k} = \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & \ddots \\ 0 & & \lambda \end{bmatrix}\]

is \(l \times l\), then

\[e^{tJ_{k}} = e^{\lambda t} \begin{bmatrix} 1 & t & \cdots & \frac{e^{\lambda t} - 1}{\lambda} \\ & 1 & \ddots & \ddots \\ & & \ddots & \frac{1}{\lambda} \\ 0 & & & 1 \end{bmatrix}.\]
The solution of the inhomogeneous IVP \( x' = Ax + b(t), \quad x(t_0) = y \) is

\[
x(t) = e^{(t-t_0)A}y + \int_{t_0}^{t} e^{(t-s)A}b(s)ds
\]

since \((e^A)^{-1} = e^{-A}\) and \(e^{A}e^{-sA} = e^{(t-s)A}\).

**Another viewpoint**

Suppose \( A \in \mathbb{C}^{n \times n} \) is a constant diagonalizable matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \) and linearly independent eigenvectors \( v_1, \ldots, v_n \). Then \( \varphi_j(t) \equiv e^{\lambda_j t}v_j \) is a solution of (LHC) \( x' = Ax \) since

\[
\varphi'_j = \frac{d}{dt}(e^{\lambda_j t}v_j) = \lambda_j e^{\lambda_j t}v_j = e^{\lambda_j t}(\lambda_j v_j) \\
= e^{\lambda_j t}Av_j = A(e^{\lambda_j t}v_j) = A\varphi_j.
\]

Clearly \( \varphi_1, \ldots, \varphi_n \) are linearly independent at \( t = 0 \) as \( \varphi_j(0) = v_j \). Thus

\[
\Phi(t) = [\varphi_1(t) \varphi_2(t) \cdots \varphi_n(t)]
\]

is a F.M. for (LHC). So the general solution of (LHC) (for diagonalizable \( A \)) is \( \Phi(t)c = c_1e^{\lambda_1 t}v_1 + \cdots + c_n e^{\lambda_n t}v_n \) for arbitrary scalars \( c_1, \ldots, c_n \).

**Remark on Exponentials**

Let \( B(t) \) be a \( C^1 \) \( n \times n \) matrix function of \( t \), and let \( A(t) = B'(t) \). Then

\[
\frac{d}{dt}(e^{B(t)}) = \frac{d}{dt}(I + B + \frac{1}{2!}B \cdot B + \frac{1}{3!}B \cdot B \cdot B + \cdots) \\
= A + \frac{1}{2!}(AB + BA) + \frac{1}{3!}(AB^2 + BAB + B^2A) + \cdots.
\]

Now, if for each \( t \), \( A(t) \) and \( B(t) \) commute, then

\[
\frac{d}{dt}(e^{B(t)}) = A \left( I + B + \frac{1}{2!}B^2 + \cdots \right) = B'(t)e^{B(t)}.
\]

Now suppose we start with a continuous \( n \times n \) matrix function \( A(t) \), and for some \( t_0 \), we define \( B(t) = \int_{t_0}^{t} A(s)ds \), so \( B'(t) = A(t) \). Suppose in addition that \( A(t) \) and \( B(t) \) commute for all \( t \). Then \( \Phi(t) = \exp \left( \int_{t_0}^{t} A(s)ds \right) \) is the F.M. for (LH) \( x' = A(t)x \), normalized at \( t_0 \), since \( \Phi(t_0) = I \) and \( \Phi'(t) = A(t)\Phi(t) \) as above.

**Remark.** A sufficient (but not necessary) condition guaranteeing that \( A(t) \) and \( \int_{t_0}^{t} A(s)ds \) commute is that \( A(t) \) and \( A(s) \) commute for all \( t, s \).
Application to Nonlinear Solution Operator

Consider the nonlinear DE $x' = f(t, x)$ where $f$ is $C^1$, and let $S^t_\tau$ denote the solution operator. For a fixed $\tau$, let $x(t, y)$ denote the solution of the IVP $x' = f(t, x)$, $x(\tau) = y$. The equation of variation for the $n \times n$ Jacobian matrix $D_y x$ is

$$\frac{d}{dt} (D_y x(t, y)) = (D_x f(t, x(t, y))) (D_y x(t, y)),$$

and thus

$$\frac{d}{dt} (\det (D_y x(t, y))) = \text{tr} (D_x f(t, x(t, y))) \det (D_y x(t, y)).$$

This relation will be used and interpreted below. Solving, one obtains

$$\det (D_y x(t, y)) = \det (D_y x(\tau, y)) \exp \left( \int_{\tau}^{t} \text{tr} (D_x f(s, x(s, y))) ds \right)$$

$$= \exp \left( \int_{\tau}^{t} \text{tr} (D_x f(s, x(s, y))) ds \right),$$

since

$$D_y x(\tau, y) = D_y y = I.$$

In particular, $\det (D_y x(t, y)) \neq 0$, so $D_y x(t, y)$ is invertible. For $\tau$ and $t$ fixed, $D_y x(t, y) = D_y S^t_\tau$, so we have demonstrated again that $D_y S^t_\tau$ is invertible at each $y$.

Rate of Change of Volume in a Flow

Consider an autonomous system $x' = f(x)$, where $f$ is $C^1$ and $F = \mathbb{R}$, so $x \in \mathbb{R}^n$. Fix $t_0$, and view the family of IVPs

$$x' = f(x), \quad x(t_0) = y$$

for $y$ in an open set $U \subset \mathbb{R}^n$ as a flow: at the initial time $t_0$, there is a particle at each point $y \in U$; that particle’s location at time $t \geq t_0$ is given by $x(t, y)$, where $x(t, y)$ is the solution of the IVP $x' = f(x)$, $x(t_0) = y$ (e.g., $f$ can be thought of as a steady-state velocity field).

For $t \geq t_0$, let $U(t) = \{x(t, y) : y \in U\}$. Then $U(t) = S^t_{t_0}(U)$ and $S^t_{t_0} : U \rightarrow U(t)$ is (for fixed $t$) a $C^1$ diffeomorphism (i.e., for fixed $t$, the map $y \mapsto x(t, y)$ is a $C^1$ diffeomorphism on $U$). In particular, $\det D_y x(t, y)$ never vanishes. Assuming, in addition, that $U$ is connected, $\det D_y x(t, y)$ must either be always positive or always negative; since $\det D_y x(t_0, y) = \det I = 1 > 0$, $\det D_y x(t, y)$ is always $> 0$.

Now the volume $\text{vol}(U(t))$ satisfies

$$\text{vol}(U(t)) = \int_{U(t)} 1 \, dx = \int_{U} |\det D_y x(t, y)| dy = \int_{U} \det D_y x(t, y) dy.$$
Assuming differentiation under the integral sign is justified (e.g., if \( U \) is contained in a compact set \( K \) and \( S^t_{t_0} \) can be extended to \( y \in K \)), and using the relation derived in the previous section,

\[
\frac{d}{dt} \left( \text{vol}(U(t)) \right) = \int_U \frac{d}{dt} (\det D_y x(t,y)) \, dy = \int_U \text{div}(x(t,y)) \det D_y x(t,y) \, dy
\]

\[
= \int_U \text{div}f(x) \, dx,
\]

where the divergence of \( f \) is by definition

\[
\text{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} + \cdots + \frac{\partial f_n}{\partial x_n} = \text{tr} \left( D_x f(x) \right).
\]

Thus the rate of change of the volume of \( U(t) \) is the integral of the divergence of \( f \) over \( U(t) \).

In particular, if \( \text{div}f(x) \equiv 0 \), then \( \frac{d}{dt} (\text{vol}(U(t))) = 0 \), and volume is conserved.

**Remark.** The same argument applies when \( f = f(t,x) \) depends on \( t \) as well: just replace \( \text{div}f(x) \) by \( \text{div}_x f(t,x) \), the divergence of \( f \) (with respect to \( x \)):

\[
\text{div}_x f(t,x) = \left( \frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n} \right) \bigg|_{(t,x)}.
\]

## Linear Systems with Periodic Coefficients

Let \( A : \mathbb{R} \to \mathbb{C}^{n \times n} \) be continuous and periodic with period \( \omega > 0 \):

\[
(\forall t \in \mathbb{R}) \quad A(t + \omega) = A(t).
\]

Note that in this case we take the scalar field to be \( F = \mathbb{C} \). Consider the periodic linear homogeneous system

\[
(\text{PLH}) \quad x' = A(t)x, \quad t \in \mathbb{R}.
\]

All solutions exist for all time \( t \in \mathbb{R} \) because the system is linear and \( A \) is defined and continuous for \( t \in \mathbb{R} \).

**Lemma.** If \( \Phi(t) \) is a F.M. for (PLH), then so also is \( \Psi(t) \equiv \Phi(t + \omega) \).

**Proof.** For each \( t \), \( \Psi(t) = \Phi'(t + \omega) = A(t + \omega)\Phi(t + \omega) = A(t)\Psi(t) \), so \( \Psi(t) \) is a matrix solution of (PLH). \( \square \)

**Theorem.** To each F.M. \( \Phi(t) \) for (PLH), there exists an invertible periodic \( C^1 \) matrix function \( P : \mathbb{R} \to \mathbb{C}^{n \times n} \) and a constant matrix \( R \in \mathbb{C}^{n \times n} \) for which \( \Phi(t) = P(t)e^{tR} \).

**Proof.** By the lemma, there is an invertible matrix \( C \in \mathbb{C}^{n \times n} \) such that \( \Phi(t + \omega) = \Phi(t)C \). Since \( C \) is invertible, it has a logarithm, i.e. there exists a matrix \( W \in \mathbb{C}^{n \times n} \) such that \( e^W = C \). Let \( R = \frac{1}{\omega}W \). Then \( C = e^{\omega R} \). Define \( P(t) = \Phi(t)e^{-tR} \). Then \( P(t) \) is invertible for all \( t \), \( P(t) \) is \( C^1 \), and \( \Phi(t) = P(t)e^{tR} \). Finally,

\[
P(t + \omega) = \Phi(t + \omega)e^{-(t + \omega)R} = \Phi(t)e^{-tR}e^{-\omega R} = \Phi(t)e^{-tR} = P(t),
\]

so \( P(t) \) is periodic. \( \square \)
Linear Scalar \( n^{\text{th}} \)-order ODEs

Let \( I \equiv [a, b] \) be an interval in \( \mathbb{R} \), and suppose \( a_j(t) \) are in \( C(I, \mathbb{F}) \) for \( j = 0, 1, \ldots, n \), with \( a_n(t) \neq 0 \) \( \forall t \in I \). Consider the \( n^{\text{th}} \)-order linear differential operator \( L : C^n(I) \to C(I) \) given by

\[
Lu = a_n(t) \frac{d^n u}{dt^n} + \cdots + a_1(t) \frac{du}{dt} + a_0(t)u,
\]

and the \( n^{\text{th}} \)-order homogeneous equation (nLH) \( Lu = 0 \), \( t \in I \). Consider the equivalent \( n \times n \) first-order system (LH) \( x' = A(t)x \), \( t \in I \), where

\[
A(t) = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
-\frac{a_n}{a_0} & 0 & \cdots & a_{n-1} \\
& \ddots & \ddots & \ddots \\
& & -\frac{a_n}{a_0} & 0 & 1
\end{bmatrix}
\text{ and } x = \begin{bmatrix}
u \\
u' \\
\vdots \\
u^{(n-1)}
\end{bmatrix} \in \mathbb{F}^n.
\]

Fix \( t_0 \in I \). Appropriate initial conditions for (nLH) are

\[
\begin{bmatrix}
u(t_0) \\
u'(t_0) \\
\vdots \\
u^{(n-1)}(t_0)
\end{bmatrix} = x(t_0) = \zeta \equiv \begin{bmatrix}\zeta_1 \\
\zeta_2 \\
\vdots \\
\zeta_n
\end{bmatrix}.
\]

Recall that \( u \) is a \( C^n \) solution of (nLH) if and only if \( x \) is a \( C^1 \) solution of (LH), with a similar equivalence between associated IVP’s. If \( \Phi(t) \) is a F.M. for (LH), with \( A(t) \) as given above, then \( \Phi(t) \) has the form

\[
\Phi = \begin{bmatrix}
\varphi_1 & \varphi_2 & \cdots & \varphi_n \\
\varphi_1' & \varphi_2' & \cdots & \varphi_n' \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)}
\end{bmatrix},
\]

where each \( \varphi_j(t) \) satisfies (nLH).

**Definition.** If \( \varphi_1(t), \ldots, \varphi_n(t) \) are solutions of (nLH), then the *Wronskian* of \( \varphi_1, \ldots, \varphi_n \) (a scalar function of \( t \)) is defined to be

\[
W(\varphi_1, \ldots, \varphi_n)(t) = \det \begin{bmatrix}
\varphi_1(t) & \varphi_2(t) & \cdots & \varphi_n(t) \\
\varphi_1'(t) & \varphi_2'(t) & \cdots & \varphi_n'(t) \\
\vdots & \vdots & \ddots & \vdots \\
\varphi_1^{(n-1)}(t) & \varphi_2^{(n-1)}(t) & \cdots & \varphi_n^{(n-1)}(t)
\end{bmatrix} (= \det \Phi(t)).
\]

Since \( \Phi(t) \) is a matrix solution of (LH), we know

\[
det (\Phi(t)) = det (\Phi(t_0)) \exp \int_{t_0}^t \text{tr} (A(s)) ds,
\]
so
\[ W(\varphi_1, \ldots, \varphi_n)(t) = W(\varphi_1, \ldots, \varphi_n)(t_0) \exp \int_{t_0}^{t} -\frac{a_{n-1}(s)}{a_n(s)} \, ds. \]

In particular, for solutions \( \varphi_1, \ldots, \varphi_n \) of (nLH),

\[ W(\varphi_1, \ldots, \varphi_n)(t) \equiv 0 \text{ on } I, \text{ or } (\forall t \in I) \quad W(\varphi_1, \ldots, \varphi_n)(t) \neq 0. \]

**Theorem.** Let \( \varphi_1, \ldots, \varphi_n \) be \( n \) solutions of (nLH) \( Lu = 0 \). Then they are linearly independent on \( I \) (i.e., as elements of \( C^n(I) \)) if and only if \( W(\varphi_1, \ldots, \varphi_n)(t) \neq 0 \) on \( I \).

**Proof.** If \( \varphi_1, \ldots, \varphi_n \) are linearly dependent in \( C^n(I) \), then there exist scalars \( c_1, \ldots, c_n \) such that

\[ c_1 \varphi_1(t) + \cdots + c_n \varphi_n(t) \equiv 0 \text{ on } I, \quad \text{with } c \equiv \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \neq 0; \]

thus \( \Phi(t)c = 0 \) on \( I \), so \( W(\varphi_1, \ldots, \varphi_n)(t) = \det \Phi(t) = 0 \) on \( I \). Conversely, if \( \det \Phi(t) = 0 \) on \( I \), then the solutions

\[ \begin{bmatrix} \varphi_1 \\ \vdots \\ \varphi_1^{(n-1)} \end{bmatrix}, \ldots, \begin{bmatrix} \varphi_n \\ \vdots \\ \varphi_n^{(n-1)} \end{bmatrix} \]

of (LH) are linearly dependent (as elements of \( C^1(I, \mathbb{F}^n) \)), so there exist scalars \( c_1, \ldots, c_n \) such that

\[ c_1 \begin{bmatrix} \varphi_1(t) \\ \vdots \\ \varphi_1(t) \end{bmatrix} + \cdots + c_n \begin{bmatrix} \varphi_n(t) \\ \vdots \\ \varphi_n(t) \end{bmatrix} \equiv 0 \text{ on } I, \]

where not all \( c_j = 0 \). In particular, \( c_1 \varphi_1(t) + \cdots + c_n \varphi_n(t) \equiv 0 \text{ on } I \), so \( \varphi_1, \ldots, \varphi_n \) are linearly dependent in \( C^n(I) \). \( \square \)

**Corollary.** The dimension of the vector space of solutions of (nLH) (a subspace of \( C^n(I) \)) is \( n \), i.e., \( \dim \mathcal{N}(L) = n \), where \( \mathcal{N}(L) \) denotes the null space of \( L : C^n(I) \to C(I) \).

The differential operator \( L \) (normalized so that \( a_n(t) \equiv 1 \)) is itself determined by \( n \) linearly independent solutions of (nLH) \( Lu = 0 \):

**Fact.** Suppose \( \varphi_1, \ldots, \varphi_n(t) \in C^n(I) \) with \( W(\varphi_1, \ldots, \varphi_n)(t) \neq 0 \) (\( \forall t \in I \)). Then there exists a unique \( n^{th} \) order linear differential operator

\[ L = \frac{d^n}{dt^n} + a_{n-1}(t) \frac{d^{n-1}}{dt^{n-1}} + \cdots + a_1(t) \frac{d}{dt} + a_0(t) \]

(with \( a_n(t) \equiv 1 \) and each \( a_j(t) \in C(I) \)) for which \( \varphi_1, \ldots, \varphi_n \) form a fundamental set of solutions of (nLH) \( Lu = 0 \), namely,

\[ Lu = \frac{W(\varphi_1, \ldots, \varphi_n, u)}{W(\varphi_1, \ldots, \varphi_n)}. \]
Ordinary Differential Equations

where

\[ W(\varphi_1, \ldots, \varphi_n, u) = \det \begin{bmatrix} \varphi_1 & \cdots & \varphi_n & u \\ \varphi'_1 & \cdots & \varphi'_n & u' \\ \vdots & \ddots & \vdots & \vdots \\ \varphi^{(n)}_1 & \cdots & \varphi^{(n)}_n & u^{(n)} \end{bmatrix} \]

Sketch. In this formula for \( Lu \), expanding the determinant in the last column shows that \( L \) is an \( n^{\text{th}} \) order linear differential operator with continuous coefficients \( a_j(t) \) and \( a_n(t) \equiv 1 \). Clearly \( \varphi_1, \ldots, \varphi_n \) are solutions of \( Lu = 0 \). For uniqueness (with \( a_n(t) \equiv 1 \)), note that if \( \varphi_1, \ldots, \varphi_n \) are linearly independent solutions of \( Lu = 0 \) for some \( L \), then

\[
\Phi^T(t) \begin{bmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_{n-1}(t) \end{bmatrix} = -\begin{bmatrix} \varphi_1^{(n)}(t) \\ \vdots \\ \varphi_n^{(n)}(t) \end{bmatrix}.
\]

Since \( W(\varphi_1, \ldots, \varphi_n) \neq 0 \ (\forall t \in I) \), \( \Phi(t) \) is invertible \( \forall t \in I \), so

\[
\left[ \begin{array}{c} a_0(t) \\ \vdots \\ a_{n-1}(t) \end{array} \right] = -(\Phi^T)^{-1}(t) \begin{bmatrix} \varphi_1^{(n)}(t) \\ \vdots \\ \varphi_n^{(n)}(t) \end{bmatrix}
\]

is uniquely determined by \( \varphi_1, \ldots, \varphi_n \).

Remark. A first-order system (LH) \( x' = A(t)x \) is uniquely determined by any F.M. \( \Phi(t) \). Since \( \Phi'(t) = A(t)\Phi(t) \), \( A(t) = \Phi'(t)\Phi^{-1}(t) \).

Linear Inhomogeneous \( n^{\text{th}} \)-order scalar equations

For simplicity, normalize the coefficients \( a_j(t) \) so that \( a_n(t) \equiv 1 \) in \( L \). Consider

\[
(\text{nLI}) \quad Lu = u^{(n)} + a_{n-1}(t)u^{(n-1)} + \cdots + a_0(t)u = \beta(t).
\]

Let

\[
x = \begin{bmatrix} u \\ u' \\ \vdots \\ u^{(n-1)} \end{bmatrix}, \quad b(t) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \beta(t) \end{bmatrix}, \quad \text{and} \quad A(t) = \begin{bmatrix} 0 & 1 & \cdots & \cdots & 1 \\ \vdots & \ddots & \ddots & & \vdots \\ -a_0 & \cdots & -a_{n-1} \end{bmatrix};
\]

then \( x(t) \) satisfies (LI) \( x' = A(t)x + b(t) \). We can apply our results for (LI) to obtain expressions for solutions of (nLI).

Theorem. If \( \varphi_1, \ldots, \varphi_n \) is a fundamental set of solutions of (nLI) \( Lu = 0 \), then the solution \( \psi(t) \) of (nLI) \( Lu = \beta(t) \) with initial condition \( u^{(k)}(t_0) = \zeta_{k+1} \ (k = 0, \ldots, n-1) \) is

\[
\psi(t) = \varphi(t) + \sum_{k=1}^n \varphi_k(t) \int_{t_0}^t \frac{W_k(\varphi_1, \ldots, \varphi_n)}{W(\varphi_1, \ldots, \varphi_n)} \beta(s) ds
\]
where $\varphi(t)$ is the solution of (nLH) with the same initial condition at $t_0$, and $W_k$ is the determinant of the matrix obtained from

$$
\Phi(t) = \begin{bmatrix}
\varphi_1 & \cdots & \varphi_n \\
\varphi'_1 & \cdots & \varphi'_n \\
\vdots & & \vdots \\
\varphi^{(n-1)}_1 & \cdots & \varphi^{(n-1)}_n
\end{bmatrix}
$$

by replacing the $k^{th}$ column of $\Phi(t)$ by the $n$-th unit coordinate vector $e_n$.

**Proof.** We know

$$
x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds,
$$

where $x_0 = [\zeta_1, \cdots, \zeta_n]^T$ and $b(s) = [0 \cdots \beta(s)]^T$, solves the IVP $x' = A(t)x$, $x(t_0) = x_0$. The first component of $x(t)$ is $\psi(t)$, and the first component of $\Phi(t)\Phi^{-1}(t_0)x_0$ is the solution $\varphi(t)$ of (nLH) described above. By Cramer’s Rule, the $k^{th}$ component of $\Phi^{-1}(s)e_n$ is

$$
\frac{W_k(\varphi_1, \cdots, \varphi_n)(s)}{W(\varphi_1, \cdots, \varphi_n)(s)}.
$$

Thus the first component of $\Phi(t) \int_{t_0}^{t} \Phi^{-1}(s)b(s)ds$ is

$$
[\varphi_1(t) \cdots \varphi_n(t)] \int_{t_0}^{t} \Phi^{-1}(s)e_n\beta(s)ds = \sum_{k=1}^{n} \varphi_k(t) \int_{t_0}^{t} \frac{W_k(\varphi_1, \cdots, \varphi_n)(s)}{W(\varphi_1, \cdots, \varphi_n)(s)} \beta(s)ds.
$$

□

**Linear $n^{th}$-order scalar equations with constant coefficients**

For simplicity, take $a_n = 1$ and $\mathbb{F} = \mathbb{C}$. Consider

$$
Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_0 u,
$$

where $a_0, \ldots, a_{n-1}$ are constants. Then

$$
A = \begin{bmatrix}
0 & 1 & & \\
& \ddots & \ddots & \\
&& 0 & 1 \\
-a_0 & \cdots & -a_{n-1}
\end{bmatrix}
$$

has characteristic polynomial

$$
p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1 \lambda + a_0.
$$
Moreover, since \( A \) is a companion matrix, each distinct eigenvalue of \( A \) has only one Jordan block in the Jordan form of \( A \). Indeed, for any \( \lambda \),

\[
A - \lambda I = \begin{bmatrix} -\lambda & 1 & & 0 \\ & \ddots & \ddots & \vdots \\ & & -a_0 & 1 \\ & & & & (-a_{n-1} - \lambda) \end{bmatrix}
\]

has rank \( \geq n - 1 \), so the geometric multiplicity of each eigenvalue is \( 1 = \dim(\mathcal{N}(A - \lambda I)) \).

Now if \( \lambda_k \) is a root of \( p(\lambda) \) having multiplicity \( m_k \) (as a root of \( p(\lambda) \)), then terms of the form \( t^j e^{\lambda_k t} \) for \( 0 \leq j \leq m_k - 1 \) appear in elements of \( e^{tJ} \) (where \( P^{-1}AP = J \) is in Jordan form), and thus also appear in \( e^{tA} = Pe^{tJ}P^{-1} \), the F.M. for (LH) \( x' = Ax \), normalized at 0. This explains the well-known result:

**Theorem.** Let \( \lambda_1, \ldots, \lambda_s \) be the distinct roots of \( p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0 = 0 \), and suppose \( \lambda_k \) has multiplicity \( m_k \) for \( 1 \leq k \leq s \). Then a fundamental set of solutions of

\[
Lu = u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_0u = 0,
\]

where \( a_k \in \mathbb{C} \), is

\[
\{t^j e^{\lambda_k t} : 1 \leq k \leq s, \ 0 \leq j \leq m_k - 1 \}.
\]

The standard proof is to show that these functions are linearly independent and then plug in and verify that they are solutions: write

\[
L = \left( \frac{d}{dt} - \lambda_1 \right)^{m_1} \cdots \left( \frac{d}{dt} - \lambda_s \right)^{m_s},
\]

and use

\[
\left( \frac{d}{dt} - \lambda_k \right)^{m_k} (t^j e^{\lambda_k t}) = 0 \quad \text{for} \quad 0 \leq j \leq m_k - 1.
\]