Fourier Series

Let \( \{ e_j : 1 \leq j \leq n \} \) be the standard basis in \( \mathbb{R}^n \). We say \( f : \mathbb{R}^n \to \mathbb{C} \) is \( 2\pi \)-periodic in each variable if
\[
f(x + 2\pi e_j) = f(x) \quad \forall x \in \mathbb{R}^n, \; 1 \leq j \leq n.
\]
We can identify \( 2\pi \)-periodic functions with functions on a torus. Let \( S^1 = \{ e^{i\theta} : \theta \in \mathbb{R} \} \subset \mathbb{C} \), and \( T^n = S^1 \times \cdots \times S^1 \subset \mathbb{C}^n \). To each function \( \tilde{\phi} : T^n \to \mathbb{C} \) we can identify a \( 2\pi \)-periodic function \( \phi : \mathbb{R}^n \to \mathbb{C} \) by \( \phi(x_1, \ldots, x_n) = \tilde{\phi}(e^{ix_1}, \ldots, e^{ix_n}) \). Conversely, each \( 2\pi \)-periodic function \( \phi : \mathbb{R}^n \to \mathbb{C} \) induces a unique \( \tilde{\phi} : T^n \to \mathbb{C} \) for which \( \tilde{\phi}(e^{ix_1}, \ldots, e^{ix_n}) = \phi(x_1, \ldots, x_n) \).

If \( \phi : \mathbb{R}^n \to \mathbb{C} \) is \( 2\pi \)-periodic, \( \phi \) is uniquely determined by its values \( \phi(x) \) for \( x \in [-\pi, \pi)^n \) or for \( x \in [0, 2\pi)^n \). Let \( \nu_n = (2\pi)^{-n}\lambda_n \), where \( \lambda_n \) is \( n \)-dimensional Lebesgue measure. Then \( \nu_n \) induces a measure \( \tilde{\nu}_n \) on \( T^n \) for which
\[
\int_{T^n} \tilde{\phi} \, d\tilde{\nu}_n = \int_{[0,2\pi]^n} \phi \, d\nu_n.
\]

From here on, we blur the distinction between \( \phi \) and \( \tilde{\phi} \) and between \( \nu_n \) and \( \tilde{\nu}_n \), and we will abuse these notations. Note: \( \nu_n(T^n) = \nu_n([0,2\pi]^n) = 1 \). Let \( L^p(T^n) \) denote \( L^p([0,2\pi]^n) \) with measure \( \nu_n \) (\( 1 \leq p \leq \infty \)). \( L^2(T^n) \) is a Hilbert space with inner product
\[
(\psi, \phi) = \int_{T^n} \bar{\psi}\phi \, d\nu_n = \int_{[0,2\pi]^n} \bar{\psi}\phi \, d\nu_n.
\]

**Theorem.** \( \{ e^{ix\cdot \xi} : \xi \in \mathbb{Z}^n \} \) is an orthonormal system in \( L^2(T^n) \).

**Proof.**
\[
(e^{ix\cdot \eta}, e^{ix\cdot \xi}) = \int_{[0,2\pi]^n} e^{ix\cdot(\xi-\eta)} \, d\nu_n = \begin{cases} 1, & \xi = \eta \\ 0, & \xi \neq \eta \end{cases}.
\]

**Definition.** A trigonometric polynomial is a finite linear combination of \( \{ e^{ix\cdot \xi} : \xi \in \mathbb{Z}^n \} \).

Note: since \( \{ e^{ix\cdot \xi}, e^{-ix\cdot \xi} \} \) and \( \{ \cos(x \cdot \xi), \sin(x \cdot \xi) \} \) span the same two-dimensional space, we could use sines and cosines as our basic functions.

**Definition.** \( C(T^n) \) is the space of all continuous \( 2\pi \)-periodic functions \( \phi : \mathbb{R}^n \to \mathbb{C} \). Note that \( C(T^n) \subsetneq C([0,2\pi]^n) \).

We will use the uniform norm \( \| \phi \|_u = \sup_x |\phi(x)| \) on \( C(T^n) \). \( C^k(T^n) \) (for \( k \geq 0, k \in \mathbb{Z} \)) is the space of all \( C^k \) \( 2\pi \)-periodic functions \( \phi : \mathbb{R}^n \to \mathbb{C} \). Again \( C^k(T^n) \subsetneq C^k([0,2\pi]^n) \). We will use the norm \( \| \phi \|_{C^k} = \sum_{|\alpha| \leq k} \| \partial^\alpha \phi \|_u \).
Fourier Coefficients

For $f \in L^1(T^n)$, define
\[
\hat{f}(\xi) = \int_{T^n} e^{-ix \cdot \xi} f(x) d\nu_n(x)
\]
for $\xi \in \mathbb{Z}^n$. Then $|\hat{f}(\xi)| \leq \|f\|_1$. We regard $\hat{f}$ as a map $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$; then $\hat{f} \in l^\infty(\mathbb{Z}^n)$ and $\|\hat{f}\|_\infty \leq \|f\|_1$. Since $\nu(T^n) = 1$, we have $L^2(T^n) \subset L^1(T^n)$ and $\|f\|_1 \leq \|f\|_2$. For $f \in L^2(T^n)$, $\hat{f}$ can be expressed as an inner product: $\hat{f}(\xi) = (e^{ix \cdot \xi}, f)$.

The Fourier series of $f$ is the formal series $\sum_{\xi \in \mathbb{Z}^n} \hat{f}(\xi) e^{ix \cdot \xi}$. We will study in what sense this series converges to $f$. A key role is played by what is called a summability kernel; this is a sequence $Q_k$ with properties (1), (2), (3), (4) below. Define
\[
Q_k(x) = a_k^{-1} \left( \prod_{j=1}^n \frac{1}{2}(1 + \cos x_j) \right)^k
\]
where
\[
a_k = \int_{T^n} \left( \prod_{j=1}^n \frac{1}{2}(1 + \cos x_j) \right)^k d\nu_n(x).
\]

Lemma.

(1) $Q_k$ is a trigonometric polynomial.

(2) $Q_k(x) \geq 0$

(3) $\int_{T^n} Q_k(x) d\nu_n(x) = 1$.

(4) For $0 < \delta < \pi$, set
\[
\eta_k(\delta) = \max \{Q_k(x) : x \in [-\pi,\pi]^n \setminus (-\delta,\delta)^n\}.
\]

Then $\lim_{k \to \infty} \eta_k(\delta) = 0$.

Proof. The first three properties are clear. To prove (4), we first show that the sequence $a_k$ satisfies $a_k \geq (2(k+1))^{-n}$. In fact, since $1 + \cos x$ is non-negative on $[0,\pi]$ and concave on $[0,\frac{\pi}{2}]$, we have
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \frac{1}{2}(1 + \cos x) \right)^k dx \geq \frac{1}{\pi} \int_0^{\pi/2} \left( \frac{1}{2}(1 + \cos x) \right)^k dx = \int_0^{\pi/2} \left( 1 - \frac{x}{\pi} \right)^k dx = (k+1)^{-1}(1 - 2^{-k-1}) \geq (2(k+1))^{-1}.
\]

Now if $x \in [-\pi,\pi]^n \setminus (-\delta,\delta)^n$, then
\[
Q_k(x) \leq a_k^{-1} \left( \frac{1}{2}(1 + \cos \delta) \right)^k \leq (2(k+1))^n \left( \frac{1}{2}(1 + \cos \delta) \right)^k \to 0
\]
Theorem. Given \( f \in C(T^n) \), let

\[
p_k(x) = (f * Q_k)(x) = \int_{T^n} f(x-y)Q_k(y)d\nu_n(y).
\]

Then \( p_k \) is a trigonometric polynomial, and \( \|p_k - f\|_u \to 0 \) as \( k \to \infty \).

Proof. Since \( \hat{Q}_k(\xi) = 0 \) for \( |\xi| \) sufficiently large,

\[
p_k(x) = \int_{T^n} f(y)Q_k(x-y)d\nu_n(y) = \int_{T^n} f(y)\sum_{\xi} \hat{Q}_k(\xi)e^{i(x-y)\cdot\xi}d\nu_n(y)
\]

is a trigonometric polynomial. Given \( \epsilon > 0 \), the uniform continuity of \( f \) implies that there is \( \delta > 0 \) such that

\[
|x_1 - x_2|_\infty < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon.
\]

Then

\[
|p_k(x) - f(x)| = \left|\int_{T^n} (f(x-y) - f(x))Q_k(y)d\nu_n(y)\right|
\]

\[
\leq \int_{T^n} |f(x-y) - f(x)|Q_k(y)d\nu_n(y) = I_1 + I_2,
\]

where \( I_1 \) is the integral over \( y \in (-\delta, \delta)^n \) and \( I_2 \) is the integral over \( y \in [-\pi, \pi]^n \setminus (-\delta, \delta)^n \). Now

\[
I_1 \leq \int_{(-\delta, \delta)^n} \epsilon Q_k(y)d\nu_n(y) \leq \epsilon, \quad \text{and}
\]

\[
I_2 \leq \int_{[-\pi, \pi]^n \setminus (-\delta, \delta)^n} 2\|f\|_u \eta_k(\delta)d\nu_n \leq 2\|f\|_u \eta_k(\delta) < \epsilon
\]

for \( k \) sufficiently large, so the result follows. \( \square \)

Corollary. Trigonometric polynomials are dense in \( C(T^n) \).

Remark. The proof of the Theorem above used only the properties (1)–(4) of the \( Q_k \). Therefore the same result holds for any summability kernel. Another example for \( n = 1 \) is the Féjer kernel, which is given by

\[
F_k(x) = (k+1)^{-1}\frac{\sin^2 \left( \frac{k}{2}(k+1)x \right)}{\sin^2 \left( \frac{x}{2} \right)} = \sum_{k=1}^{k} \left( 1 - \frac{|\xi|}{k+1} \right) e^{ix\xi}.
\]

If we define \( S_k(f) = \sum_{\xi=-k}^{k} \hat{f}(\xi)e^{ix\xi} \) and \( \sigma_k(f) = (k+1)^{-1}(S_0(f) + \cdots + S_k(f)) \), then \( \sigma_k(f) = f * F_k \). It follows that for \( f \in C(T) \), \( \sigma_k(f) \to f \) uniformly. This is the classical result that the Fourier series of \( f \in C(T) \) is Cesàro summable to \( f \).
The partial sums $S_k(f)$ themselves are obtained by convolving $f$ with the “Dirichlet kernel”: $S_k(f) = f \ast D_k$, where

$$D_k(x) = \sum_{\xi=-k}^{k} e^{ix\xi} = \frac{\sin((k + \frac{1}{2})x)}{\sin(\frac{1}{2}x)}.$$  

The Dirichlet kernel however, is not a summability kernel: $D_k$ is not nonnegative (not horrible), and more crucially it does not satisfy condition (4) of a summability kernel. This is the reason that pointwise convergence of Fourier series is a delicate matter.

**Corollary.** Trigonometric polynomials are dense in $L^2(T^n)$.

**Proof.** Given $f \in L^2(T^n)$ and $\epsilon > 0$, there exists $g \in C(T^n)$ such that $\|f - g\|_2 < \frac{1}{2}\epsilon$ and there exists a trigonometric polynomial $p$ such that $\|p - g\|_2 < \frac{1}{2}\epsilon$, so since $\nu_n(T^n) = 1$,

$$\|f - p\|_2 \leq \|f - g\|_2 + \|g - p\|_2 \leq \|f - g\|_2 + \|g - p\|_1 < \epsilon.$$  

□

**Corollary.** $\{e^{ix\xi} : \xi \in \mathbb{Z}^n\}$ is a complete orthonormal system in $L^2(T^n)$.

Hence if $f \in L^2(T^n)$, the Fourier series of $f$ (any arrangement) converges to $f$ in $L^2$. Also, the map $\mathcal{F} : L^2(T^n) \rightarrow l^2(\mathbb{Z}^n)$ given by $f \mapsto \hat{f}$ is a Hilbert space isomorphism and $\|\hat{f}\|_{l^2(\mathbb{Z}^n)} = \|f\|_{L^2(T^n)}$.

**Theorem.** If $f \in L^1(T^n)$, then $p_k \rightarrow f$ in $L^1(T^n)$, where $p_k = f \ast Q_k$ and $Q_k$ is as above.

**Proof.** The proof is similar to the proof of the Theorem above, except we use continuity of translation in $L^1$ instead of uniform continuity. Given $\epsilon > 0$, choose $\delta \ni \|f(x-\alpha) - f(x)\|_1 < \epsilon$ whenever $|\alpha|_\infty < \delta$. By Fubini,

$$\int_{T^n} |p_k(x) - f(x)|d\nu_n(x) \leq \int_{T^n} \left[ Q_k(y) \int_{T^n} |f(x-y) - f(x)|d\nu_n(x) \right] d\nu_n(y) = I_1 + I_2,$$

and

$$I_1 \leq \int Q_k(y) \epsilon d\nu_n(y) = \epsilon$$

$$I_2 \leq 2\|f\|_1 \eta_k(\delta) \rightarrow 0 \text{ as } k \rightarrow \infty.$$  

□

**Corollary.** (Uniqueness Theorem). If $f \in L^1(T^n)$ and $(\forall \xi \in \mathbb{Z}^n)\hat{f}(\xi) = 0$, then $f = 0$ a.e. Thus if $f, g \in L^1(T^n)$ and $\hat{f} \equiv \hat{g}$, then $f = g$ a.e.

**Proof.** If $\hat{f} \equiv 0$, then $p_k(x) = \sum_\xi \hat{f}(\xi)\hat{Q}_k(\xi)e^{ix\xi} = 0$, and $p_k \rightarrow f$ in $L^1$. □

**Theorem.** (Riemann-Lebesgue Lemma). If $f \in L^1(T^n)$, then $\hat{f}(\xi) \rightarrow 0$ as $|\xi| \rightarrow \infty$.

**Proof.** This follows from the analogous result for the Fourier transform, which will be proved later. The statement for the Fourier transform is that if $f \in L^1(\mathbb{R}^n)$, then $\int_{\mathbb{R}^n} e^{-ix\cdot \xi}f(x)dx \rightarrow 0$ as $|\xi| \rightarrow \infty$, $\xi \in \mathbb{R}^n$. The Fourier series version stated here follows by applying the $\mathbb{R}^n$ version to $f(x)\chi_{[-\pi,\pi]^n}(x) \in L^1(\mathbb{R}^n)$ and restricting to $\xi \in \mathbb{Z}^n$. □
Absolutely Convergent Fourier Series

**Theorem.** Suppose $f \in L^1(T^n)$ and $\hat{f} \in l^1(\mathbb{Z}^n)$. Then the Fourier series of $f$ converges absolutely and uniformly to a $g \in C(T^n)$, and $g = f$ a.e.

**Proof.** Let $g(x) = \sum_{\xi \in \mathbb{Z}^n} \hat{f}(\xi) e^{ix\xi}$. Since $\hat{f} \in l^1(\mathbb{Z}^n)$, this series converges uniformly and absolutely, and $g \in C(T^n)$. By the Dominated Convergence Theorem,

$$\hat{g}(\xi) = \int_{T^n} e^{-ix\xi} \left( \sum_{\eta \in \mathbb{Z}^n} \hat{f}(\eta) e^{ix\eta} \right) d\nu_n(x) = \sum_{\eta \in \mathbb{Z}^n} \hat{f}(\eta) \int_{T^n} e^{-ix\xi} e^{ix\eta} d\nu_n(x) = \hat{f}(\xi),$$

so $g = f$ a.e. \hfill $\square$

**Decay of Fourier Coefficients $\leftrightarrow$ Smoothness of $f$**

**Lemma.** Suppose $\alpha(\xi) \in l^1(\mathbb{Z}^n)$ and $i\xi_j \alpha(\xi) \in l^1(\mathbb{Z}^n)$. Let $f = \sum \alpha(\xi) e^{ix\xi}$ and $g = \sum i\xi_j \alpha(\xi) e^{ix\xi}$. Then $f, g \in C(T^n)$, $\partial_j f$ exists everywhere, and $\partial_j f = g$.

**Proof.** The two series of continuous functions converge absolutely and uniformly to $f$ and $g$, respectively. Since $\partial_j (\alpha(\xi) e^{ix\xi}) = i\xi_j \alpha(\xi) e^{ix\xi}$, the result follows from a standard theorem in analysis. \hfill $\square$

**Theorem.** Suppose $f \in L^1(T^n)$ and $(1 + |\xi|^m)\hat{f}(\xi) \in l^1(\mathbb{Z}^n)$ for some integer $m \geq 0$. Then the Fourier series of $f$ converges absolutely and uniformly to a $g \in C^m(T^n)$, and $f = g$ a.e.

**Proof.** By the theorem above, we only have to show that $g \in C^m(T^n)$. For each $\nu$ with $|\nu| \leq m$, $(i\xi)^\nu \hat{f}(\xi) \in l^1(\mathbb{Z}^n)$, so $\sum_i (i\xi)^\nu \hat{f}(\xi) e^{ix\xi}$ converges absolutely and uniformly to some $g_\nu \in C(T^n)$. By the Lemma and induction, $g_\nu = \partial^\nu g$. \hfill $\square$

**Theorem.** Suppose $f \in C^m(T^n)$.

(a) For $|\nu| \leq m$, $\partial^\nu \hat{f}(\xi) = (i\xi)^\nu \hat{f}(\xi)$.

(b) $(1 + |\xi|^m)\hat{f}(\xi) \in L^2(\mathbb{Z}^n)$ and $\|(1 + |\xi|^m)\hat{f}(\xi)\|_{L^2(\mathbb{Z}^n)} \leq c_{n,m} \|f\|_{C^m(T^n)}$ for some constant $c_{n,m}$ depending only on $n, m$.

(c) $|\hat{f}(\xi)| \leq c_{n,m} (1 + |\xi|)^{-m} \|f\|_{C^m(T^n)}$ for $\xi \in \mathbb{Z}^n$.

**Proof.**
(a) follows by integration by parts. Set $x' = (x_2, \ldots, x_n)$, so $x = (x_1, x')$. Then

$$\frac{\partial f}{\partial x_1}(\xi) = \int_{T_{n-1}} \left[ \int_T e^{-ix\cdot\xi} \frac{\partial f}{\partial x_1}(x) \, d\nu_1(x_1) \right] \, d\nu_{n-1}(x')$$

$$= \int_{T_{n-1}} \left[ (i\xi_1) \int_T e^{-ix\cdot\xi} f(x) \, d\nu_1(x_1) \right] \, d\nu_{n-1}(x')$$

$$= (i\xi_1) \hat{f}(\xi).$$

Iterate for higher derivatives.

(b) Part (a) gives for $1 \leq j \leq n$:

$$\|\xi_j^m \hat{f}\|_{l^2(\mathbb{Z}^n)} = \|\hat{\partial^m_{x_j} f}\|_{l^2(\mathbb{Z}^n)} = \|\partial^m_{x_j} f\|_{L^2(T^n)} \leq \|\partial^m_{x_j} f\|_u.$$

Adding gives

$$\|(1 + |\xi_1|^m + \cdots + |\xi_n|^m)|\hat{f}(\xi)||_{l^2(\mathbb{Z}^n)} \leq \|f\|_{C^m(T^n)},$$

and (b) follows.

(c) is immediate from (b) and the fact that $\|\cdot\|_\infty \leq \|\cdot\|_2$ on functions on $\mathbb{Z}^n$. \qed

In comparing the last two theorems, we see that in order to conclude that a given $f \in L^1(T^n)$ is $C^m$, it suffices to know that $(1 + |\xi|)^m \hat{f} \in l^1$, and in the other direction, if $f$ is $C^m$, then $(1 + |\xi|)^m \hat{f} \in l^2$. Thus the space of Fourier coefficients of $C^m$ functions lies between $(1 + |\xi|)^{-m} l^1$ and $(1 + |\xi|)^{-m} l^2$. So faster decay of $\hat{f}$ corresponds to more smoothness of $f$. 