L^p Spaces

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Math 555, Winter 2014

Simplest Case: $L^1(\mathbb{R}^n)$

The collection of integrable functions on $\mathbb{R}^n$ is a vector space.

Define: $$\|f\|_1 = \int |f|$$

- $\|cf\|_1 = |c|\,\|f\|_1\,, \quad c \in \mathbb{C}$
- $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$

$\|\cdot\|_1$ is a *seminorm* but not a norm: $\|f\|_1 = 0$ iff $f(x) = 0$ a.e.

Definition

$L^1(\mathbb{R}^n)$ is the vector space of equivalence classes of integrable functions on $\mathbb{R}^n$, where f is equivalent to g if $f = g$ a.e.
Then $\|\cdot\|_1$ makes $L^1(\mathbb{R}^n)$ into a normed vector space.

Theorem (Riesz-Fischer)

$L^1(\mathbb{R}^n)$ is complete under $\|\cdot\|_1$, i.e. $L^1(\mathbb{R}^n)$ is a Banach space.

Proof. Show Cauchy sequence has convergent subsequence:

WTS if $\lim_{m,n\to\infty} \|f_n - f_m\|_1 = 0\,,\ \ \exists\, f_{n_j}, f$ s.t. $\lim_{j\to\infty} \|f_{n_j} - f\|_1 = 0$

Take n_j s.t. $\|f_{n_j} - f_{n_{j-1}}\| \le 2^{-j}$, so that $\displaystyle\sum_{j=2}^{\infty} \int |f_{n_j} - f_{n_{j-1}}| < \infty\,.$

Last time: f_{n_j} converges pointwise a.e. to $f = f_1 + \sum_{j=2}^{\infty} f_{n_j} - f_{n_{j-1}}$

$$|f - f_{n_j}| \le \sum_{j=2}^{\infty} |f_{n_j} - f_{n_{j-1}}|\,, \quad \text{LDCT} \ \Rightarrow \ \lim_{j\to\infty} \int |f - f_{n_j}| = 0$$

Definition: $1 \leq p < \infty$

$L^p(\mathbb{R}^n)$ is the vector space of equivalence classes of integrable functions on $\mathbb{R}^n$, where f is equivalent to g if $f = g$ a.e., such that $\int |f|^p < \infty$. We define $\|f\|_p = \left(\int |f|^p\right)^{1/p}$.

Remarks

- $L^p(\mathbb{R}^n)$ is a vector space, since

$$|f + g|^p \leq 2^p\big(|f|^p + |g|^p\big)$$

- $\|c\, f\|_p = |c|\, \|f\|_p$, and $\|f\|_p = 0$ iff $f \equiv 0$.

Need triangle inequality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$ to conclude it's a norm.

Young's Inequality

Assume $0 < p, q < 1$, and $a, b \geq 0$

$$\text{If} \quad \frac{1}{p} + \frac{1}{q} = 1, \quad \text{then} \quad ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Follows from convexity of exp:

$$\exp\Big(\frac{x}{p} + \frac{y}{q}\Big) \leq \frac{\exp(x)}{p} + \frac{\exp(y)}{q}$$

$$\text{with} \quad x = \log(a^p), \quad y = \log(b^q)$$

Immediate consequence:

$$\int |fg| \leq \frac{1}{p}\int |f|^p + \frac{1}{q}\int |g|^q$$

Hölder's Inequality

If $f \in L^p$ and $g \in L^q$, where $1 < p, q < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, then $fg \in L^1$, and $\|fg\|_1 \le \|f\|_p \|g\|_q$.

Proof. Suffices to consider $\|f\|_p = 1$, $\|g\|_q = 1$, in which case

$$\|fg\|_1 = \int |fg| \ \le \ \frac{1}{p} \int |f|^p + \frac{1}{q} \int |g|^q \ = \ 1 \, .$$

Minkowski's Inequality

For $1 \le p < \infty$, $\|f + g\|_p \ \le \ \|f\|_p + \|g\|_p \, .$

Proof. $\int |f+g|^p \le \int |f||f+g|^{p-1} + \int |g||f+g|^{p-1}$

$$\begin{aligned} \|f+g\|_p^p \ &\le \ \Big(\|f\|_p + \|g\|_p \Big) \big\| |f+g|^{p-1} \big\|_{p/(p-1)} \\ &\le \ \Big(\|f\|_p + \|g\|_p \Big) \|f+g\|_p^{p-1} \end{aligned}$$

Theorem (Riesz-Fischer)

$L^p(\mathbb{R}^n)$ is complete under $\|\cdot\|_p$, i.e. $L^p(\mathbb{R}^n)$ is a Banach space.

Proof. Similar to $p = 1$: suppose $\|f_{n_j} - f_{n_{j-1}}\|_p \leq 2^{-j}$. Then

$$\Big\| \sum_{j=2}^{\infty} |f_{n_j} - f_{n_{j-1}}| \Big\|_p \leq \sum_{j=2}^{\infty} 2^{-j} < \infty,$$

$$\sum_{j=2}^{\infty} |f_{n_j}(x) - f_{n_{j-1}}(x)| < \infty \quad \text{for a.a. } x, \quad \text{so} \quad f_{n_j}(x) \to f(x) \text{ a.e.}$$

Definition

For any measurable set $A \subset \mathbb{R}^n$, define $\|f\|_{L^p(A)} = \left(\int_A |f|^p\right)^{1/p}$.

$L^p(A) =$ equivalency classes of measurable functions on A.

Case $p = \infty$: analogue of sup norm

For a measurable function f, set

$$\|f\|_\infty = \inf \big\{ c \,:\, |f(x)| \le c \;\text{ for a.a. } x \big\}$$

- Equivalent characterization: $\|f\|_\infty \le c$ if $|f(x)| \le c$ a.e.
- $\|\cdot\|_\infty$ is a norm on the space of equivalency classes; in particular $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$
- $p = 1\,,\; q = \infty\,,$ holds for Hölder's: $\|fg\|_1 \le \|f\|_1 \|g\|_\infty$

Theorem

$L^\infty(\mathbb{R}^n)$ is a Banach space, i.e. it is complete in the norm.

Proof. $|f_m(x) - f_n(x)| \le \|f_m - f_n\|_\infty$ except on null-set $E_{m,n}$.
Then f_m is uniformly convergent on complement of $\bigcup_{m,n} E_{m,n}$.

Dense sets in L^p, for $1 \leq p < \infty$

Theorem

Finite simple functions $g = \sum_{j=1}^{N} c_j \, \chi_{A_j}$ are dense in $L^p(\mathbb{R}^n)$.

Proof. Non-negative f are ε-close to such g by construction of integral. General $f = f_+ - f_- + i\big(\mathrm{Im} f\big)_+ - i\big(\mathrm{Im} f\big)_-$

Theorem

$C_c(\mathbb{R}^n)$ functions are dense in $L^p(\mathbb{R}^n)$.

Proof. By above, need show $\exists h \in C_c(\mathbb{R}^n)$ with $\|h - \chi_A\|_p < \varepsilon$. Depends on approximation in measure property for Lebesgue:

$$\exists \;\; \text{compact} \;\; K \subseteq A \subseteq U \;\; \text{open} \quad : \quad \lambda(U) < \lambda(K) + \epsilon\,.$$

Continuity of Translation. Define $f_y(x) = f(x-y)$.

Theorem

Suppose $1 \le p < \infty$, and $f \in L^p(\mathbb{R}^n)$. Given $\epsilon > 0\,,\ \exists\,\delta > 0$ s.t.

$$\|f - f_y\|_p < \varepsilon \quad \text{if} \quad |y| < \delta\,.$$

Proof. If $f \in C_c(\mathbb{R}^n)$, holds by uniform continuity, bounded support. General f, take $h \in C_c(\mathbb{R}^n)$ s.t. $\|f - h\|_p < \varepsilon/3$,

$$\begin{aligned} \|f - f_y\|_p &\le \|f - h\|_p + \|h - h_y\|_p + \|h_y - f_y\|_p \\ &\le \frac{\varepsilon}{3} + \|h - h_y\|_p + \frac{\varepsilon}{3} \end{aligned}$$

Fails in L^∞ : $\|\chi_{[0,1]} - \chi_{[y,y+1]}\|_\infty = 1$ for all $y \neq 0$.

Local L^p spaces

Definition

G open, say $f \in L^p_{loc}(G)$ if $\int_K |f|^p < \infty$ for each compact $K \subset G$

- $C(G) \subset L^p_{loc}(G)$.
- $L^p(G) \subsetneq L^p_{loc}(G)$, not equal since $C(G) \not\subset L^p(G)$.
- $L^p_{loc}(G)$ is a *semi-normed* vector space: semi-norms given by family $\| \cdot \|_{L^p(K)}$ for collection of compact $K \subset G$.
  If exhaust G by countable collection of K_j :

$$K_j \subset \mathrm{int}(K_{j+1}) \,, \qquad G = \bigcup_{j=1}^{\infty} K_j \,,$$

suffices to use countable family of seminorms $\| \cdot \|_{L^p(K_j)}$.

Lemma

The seminorm topology on $L^p_{loc}(G)$ is equivalent to a metric space topology, with metric

$$d(f,g) = \sum_{j=1}^{\infty} 2^{-j} \, \frac{\|f-g\|_{L^p(K_j)}}{1+\|f-g\|_{L^p(K_j)}}$$