Lecture 3: The Schwartz space

Hart Smith

Department of Mathematics
University of Washington, Seattle

Math 526, Spring 2013
Fourier transform and convolution

**Theorem**

Suppose that \( f, g \in L^1(\mathbb{R}^n) \). Then

\[
\hat{f} \ast \hat{g}(\xi) = \hat{f}(\xi) \hat{g}(\xi).
\]

**Proof.** By Tonnelli, \( e^{-ix \cdot \xi} f(x - y) g(y) \in L^1(\mathbb{R}^{2n}, d(x, y)) \).

\[
\hat{f} \ast \hat{g}(\xi) = \int e^{-ix \cdot \xi} \int f(x - y) g(y) \, dy \, dx
\]

\[
= \int \left( \int e^{-ix \cdot \xi} f(x - y) \, dx \right) g(y) \, dy
\]

\[
= \int \left( \int e^{-iz \cdot \xi} f(z) \, dz \right) e^{-iy \cdot \xi} g(y) \, dy = \hat{f}(\xi) \hat{g}(\xi)
\]

where \( z = x + y \).
The Schwartz space of functions $S(\mathbb{R}^n)$

**Definition**

A function $f : \mathbb{R}^n \to \mathbb{C}$ belongs to $S$ if $f \in C^\infty(\mathbb{R}^n)$, and for all multi-indices $\alpha$ and integers $N$ there is $C_{N,\alpha}$ such that

$$\left| \partial_x^\alpha f(x) \right| \leq C_{N,\alpha} (1 + |x|)^{-N}.$$ 

- Say that $f$ and all of its derivatives are *rapidly decreasing*.
- Equivalent condition: for all multiindices $\alpha, \beta$, $\exists C_{\alpha,\beta} < \infty$:

$$\left| x^\alpha \partial_x^\beta f(x) \right| \leq C_{\alpha,\beta}.$$ 

- There is no single norm that characterizes $S$; instead the “size” of $f$ is characterized by the countable collection of numbers $C_{N,\alpha}$ or $C_{\alpha,\beta}$.
Introduce equivalent countable families of seminorms on $S$:

$$
\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta f(x)|, \quad \|f\|_{N,\beta} = \sup_{x \in \mathbb{R}^n} |(1 + |x|)^N \partial_x^\beta f(x)|
$$

Say that a sequence $f_n \to f$ in $S$ if $\|f_n - f\|_{\alpha,\beta} \to 0$ for all $\alpha, \beta$.

Say that $T : S \to S$ is continuous if $Tf_n \to Tf$ whenever $f_n \to f$.

Introduce a metric $d(f, g)$ on $S$:

$$
d(f, g) = \sum_{\alpha, \beta} 2^{-|\alpha| - |\beta|} \frac{\|f - g\|_{\alpha,\beta}}{1 + \|f - g\|_{\alpha,\beta}}
$$

$d(f_n, f) \to 0$ if and only if $\|f_n - f\|_{\alpha,\beta} \to 0$ for all $\alpha, \beta$. 
Theorem
The space $S$ is complete in the metric $d(f, g)$.

Proof. Suppose $\{f_n\} \subset S$ is Cauchy: $\lim_{m,n \to \infty} d(f_n, f_m) = 0$.
For each fixed $\alpha, \beta$, $\{x^\alpha \partial_x^\beta f_n\}_{n=1}^\infty$ is Cauchy in uniform norm, so

$$\lim_{n \to \infty} x^\alpha \partial_x^\beta f_n = f_{\alpha, \beta} \text{ uniformly, some } f_{\alpha, \beta} \in C_0(\mathbb{R}^n).$$

Lemma
If $\{g_n\} \subset C^1(\mathbb{R}^n)$ converges uniformly to $g$, and $\partial_j g_n$ converges uniformly to $g(j)$, then $g \in C^1(\mathbb{R}^n)$, and $\partial_j g = g(j)$.

- Conclude by induction: $f := f_{0,0} \in C^\infty(\mathbb{R}^n)$, and $\partial_x^\beta f_n \to \partial_x^\beta f$ uniformly for every $\beta$.
- Easily follows that $x^\alpha \partial_x^\beta f_n \to x^\alpha \partial_x^\beta f$ uniformly, so $f \in S$, and $d(f_n, f) \to 0$. 
Lemma

\( C_c^\infty(\mathbb{R}^n) \subset S \) is dense in the metric topology.

*Proof.* Let \( \Phi(x) \in C_c^\infty \) satisfy \( \Phi(x) = \begin{cases} 1, & |x| \leq 1 \\ 0, & |x| \geq 2 \end{cases} \)

Claim: \( \| f - \Phi(R^{-1} \cdot f) \|_{\alpha,\beta} \to 0 \) as \( R \to \infty \) each \( \alpha, \beta \), if \( f \in S \).

1. \( (1 + |x|)^N (1 - \Phi(R^{-1}x))|f(x)| \leq R^{-1}(1 + |x|)^N+1 |f(x)| \)

so:

\[
\| f - \Phi(R^{-1} \cdot f) \|_{N,0} \leq R^{-1} \| f \|_{N+1,0}
\]

2. \( \partial_x^\beta \Phi(R^{-1} \cdot f) = \Phi(R^{-1} \cdot f) \partial_x^\beta f + R^{-1} \Phi'(R^{-1} \cdot f) \partial_x^{\beta-1} f + R^{-2} \ldots \)

so:

\[
\| f - \Phi(R^{-1} \cdot f) \|_{N,\beta} \leq C R^{-1} \sum_{\beta' \leq \beta} \| f \|_{N+1,\beta'}
\]
Lemma

The space $S$ maps continuously into $L^p(\mathbb{R}^n)$ for each $p \in [1, \infty]$, and the image is dense in the $L^p$ norm if $p \in [1, \infty)$.

Proof. Density follows from density of $C_c^\infty(\mathbb{R}^n)$.

For inclusion (if $p < \infty$, the case $p = \infty$ is trivial):

$$
\left( \int |f(x)|^p \, dx \right)^{1/p} = \left( \int (1 + |x|)^{-p(n+1)} \cdot (1 + |x|)^{p(n+1)} |f(x)|^p \, dx \right)^{1/p} \\
\leq \|f\|_{n+1,0} \left( \int (1 + |x|)^{-p(n+1)} \, dx \right)^{1/p} \\
\leq C_n \|f\|_{n+1,0}
$$