# Lecture 2: Convolution

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# Convolution product

If $f$, $g$ functions on $\mathbb{R}^n$, formally define

$$(f * g)(x) = \int f(x-y)\, g(y)\, dy\,.$$

**Theorem**

Suppose that $f, g \in L^1(\mathbb{R}^n)$. Then for almost all $x$, the function $f(x-y)\, g(y)$ is integrable in $y$, and

$$\int \bigl|(f * g)(x)\bigr|\, dx \le \|f\|_{L^1}\, \|g\|_{L^1}\,.$$

# Proof.

$f(x-y)\,g(y)$ is measurable on $\mathbb{R}^{2n}$, and by Tonnelli theorem:

$$\int_{\mathbb{R}^{2n}} |f(x-y)\,g(y)|\,d(x,y) = \int\left(\int |f(x-y)|\,|g(y)|\,dx\right)dy$$
$$= \left(\int |f(x)|\,dx\right)\left(\int |g(y)|\,dy\right) < \infty$$

By Fubini, $f(x-y)\,g(y) \in L^1(dy)$ for almost all $x$, so $(f*g)(x)$ is defined for almost all $x \in \mathbb{R}^n$.

$$\int |f*g|(x)\,dx = \int\left|\int f(x-y)\,g(y)\,dy\right|dx$$
$$\leq \int\int |f(x-y)|\,|g(y)|\,dy\,dx = \|f\|_{L^1}\|g\|_{L^1}$$

$(f * g)(x) = (g * f)(x)$ where the integral exists.

*Proof.* Letting $z = x - y$,

$$\int f(x-y)\, g(y)\, dy = \int f(z)\, g(x-z)\, dz$$

$f * (g * h) = (f * g) * h$

*Proof.* Where the $d(y,z)$ integral exists (which is $x$ a.e.)

$$\begin{aligned} f * (g * h)(x) &= \int\int f(x-y)\, g(y-z)\, h(z)\, dz\, dy \\ &= \int\int f(x-z-y)\, g(y)\, h(z)\, dy\, dz = (f * g) * h(x) \end{aligned}$$

where $y \to y + z$ in the second line.

## Theorem

The product $*$ turns the Banach space $L^1(\mathbb{R}^n)$ into a commutative and associative algebra, for which

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1} .$$

That is, $\left[L^1(\mathbb{R}^n), *\right]$ is a commutative *Banach algebra*.

More common usage of convolution: suppose $K(x) \in L^1(\mathbb{R}^n)$. Then the linear mapping

$$f \to K * f$$

is a bounded map on $L^1(\mathbb{R}^n)$ with operator norm $\leq \|K\|_{L^1}$ , i.e.

$$\|K * f\|_{L^1} \leq \|K\|_{L^1} \|f\|_{L^1} .$$

Call $K$ a *convolution kernel*.

# Convolution kernels on $L^p$

**Theorem**

Suppose $K \in L^1(\mathbb{R}^n)$, and $f \in L^p(\mathbb{R}^n)$ , some $p \in [1,\infty]$. Then $\int K(x-y)f(y)\,dy$ exists for almost all $x$, and

$$\|K * f\|_{L^p} \le \|K\|_{L^1}\|f\|_{L^p}\,.$$

*Proof.* For $p=\infty$, then $|K(x-y)|\,|f(y)| \le |K(x-y)|\,\|f\|_{L^\infty}$ for almost all $y$, so $K * f(x)$ exists for every $x$, and for all $x$

$$\left|\int K(x-y)\,f(y)\,dy\right| \le \int |K(x-y)|\,dy \cdot \|f\|_{L^\infty} = \|K\|_{L^1}\|f\|_{L^\infty}\,.$$

For $f \in L^p$, $1 \le p \le \infty$, write $f = f_0 + f_1$ where $f_0 \in L^\infty\,,\ f_1 \in L^1$, so

$$\int K(x-y)\,f(y)\,dy = \int K(x-y)\,\Big(f_0(y)+f_1(y)\Big)\,dy \ \text{ exists } x \text{ a.e.}$$

To see $\|K * f\|_{L^p} \leq \|K\|_{L^1}\|f\|_{L^p}$, use the integral form of Minkowski's inequality:

$$\left\| \int K(y)\, f(x-y)\, dy \right\|_{L^p(dx)} \leq \int \|K(y)\, f(x-y)\|_{L^p(dx)}\, dy$$

$$= \left( \int |K(y)|\, dy \right) \|f\|_{L^p}$$

$$= \|K\|_{L^1}\|f\|_{L^p}$$

# Example: averaging operator

Given $r > 0$, let $K_r(x) = \dfrac{1}{m(B(r,0))}\,\mathbb{1}_{B(r,0)}(x)$. Then $\|K_r\|_{L^1} = 1$, and

$$\begin{aligned} K_r * f(x) &= \frac{1}{m(B(r,0))}\int \mathbb{1}_{B(r,0)}(x-y)\,f(y)\,dy \\ &= \frac{1}{m(B(r,0))}\int_{|y-x|<r} f(y)\,dy \\ &= A_r f(x) \end{aligned}$$

So: $\|A_r f\|_{L^p} \le \|f\|_{L^p}$

*Much deeper theorem:* $Hf(x) := \sup_{r>0} A_r|f|(x)$ satisfies

$$\|Hf\|_{L^p} \le C_p\,\|f\|_{L^p}\,, \quad 1 < p \le \infty\,.$$

# Convolution and translations

**Definition**

For $y \in \mathbb{R}^n$, $f$ a function on $\mathbb{R}^n$, define $\tau_y f$ by $(\tau_y f)(x) = f(x-y)$.

- By translation invariance of measure: $\|\tau_y f\|_{L^p} = \|f\|_{L^p}$.
- By density of step functions in $L^p$ for $1 \le p < \infty$:

$$\lim_{y \to 0} \|\tau_y f - f\|_{L^p} = 0\,, \quad 1 \le p < \infty$$

- Can write $K * f$ as "*sum of translates of f*":

$$(K * f)(\,\cdot\,) = \int K(y)\, f(\,\cdot - y)\, dy = \int K(y)\, \tau_y f(\,\cdot\,)\, dy\,.$$

# Approximations to the identity

**Definition**

If $K \in L^1(\mathbb{R}^n)$, define $\quad K_r(x) = r^{-n} K(r^{-1}x)\,.$

- By change of variables:

$$\int K_r(x) = \frac{1}{r^n} \int K\Big(\frac{x}{r}\Big)\,dx = \int K(x)\,dx$$

- If $\delta > 0$, then

$$\int_{|x|>\delta} |K_r(x)|\,dx = \int_{|x|>r^{-1}\delta} |K(x)|\,dx$$

so for any fixed $\delta > 0$ :

$$\lim_{r\to 0} \int_{|x|>\delta} |K_r(x)|\,dx = 0\,.$$

# Approximations to the identity

**Theorem**

Suppose that $K \in L^1(\mathbb{R}^n)$, and $\int K(x)\,dx = 1$. If $f \in L^p(\mathbb{R}^n)$, and $1 \le p < \infty$, then $\lim_{r\to 0} \|K_r * f - f\|_{L^p} = 0$.

*Proof.* Given $\epsilon > 0$, $\exists\,\delta > 0$ such that $\|\tau_y f - f\|_{L^p} < \epsilon$ if $|y| < \delta$. Since $f(x) = \int K_r(y)\, f(x)\, dy$, write

$$\|(K_r * f) - f\|_{L^p} = \left\| \int K_r(y)\Big(\tau_y f(x) - f(x)\Big)\, dy \right\|_{L^p(dx)}$$

$$\le \int_{|y|<\delta} |K_r(y)|\, \|\tau_y f - f\|_{L^p}\, dy + \int_{|y|>\delta} |K_r(y)|\, \|\tau_y f - f\|_{L^p}\, dy$$

$$\le \epsilon \cdot \|K\|_{L^1} + 2\,\|f\|_{L^p} \int_{|y|>\delta} |K_r(y)|\, dy$$

*Both terms small if $\epsilon$ small and $r$ small.*

# Application: smooth approximations to the identity

**Lemma**

Suppose that $\Phi \in C^m(\mathbb{R}^n)$ has compact support. If $f \in L^p(\mathbb{R}^n)\,,\ p \in [1,\infty]$, then $\Phi * f \in C^m(\mathbb{R}^n)$.

*Proof.* Write $(\Phi * f)(x) = \int \Phi(x-y)\, f(y)\, dy$ and differentiate under integral sign.

**Theorem**

Suppose that $\Phi \in C_c^m(\mathbb{R}^n)$ is non-negative and $\int \Phi(y)\, dy = 1$. Then $\Phi_r * f$ is a family of $C^m$ functions, such that

$$\|\Phi_r * f\|_{L^p} \le \|f\|_{L^p}\,, \qquad \lim_{r\to 0} \|\Phi_r * f - f\|_{L^p} = 0\,.$$