Math 524, Autumn 2007, Homework 9

The following homework is due Friday, December 7.

1. A Borel measure \( \mu \) on \( \mathbb{R}^n \) is called regular if \( \mu(K) < \infty \) for all compact sets \( K \), and if for all Borel sets \( E \) we have

\[
\mu(E) = \sup\{\mu(K) : K \subseteq E, \text{ K compact}\} = \inf\{\mu(U) : U \supseteq E, \text{ U open}\}
\]

Show that if \( \mu \) and \( \nu \) are regular Borel measures, and

\[
\int \phi \, d\mu = \int \phi \, d\nu
\]

for all \( \phi \in C_c(\mathbb{R}^n) \), then \( \mu = \nu \). [Consider functions \( \phi = 1 \) on \( K \) and \( \phi = 0 \) on \( U^c \).]

2. If \( f \in L^1(\mathbb{R}, dx) \), show that

\[
\int_{x_1 < x_2 < \cdots < x_n} f(x_1)f(x_2) \cdots f(x_n) \, dx_1 \, dx_2 \cdots dx_n = \frac{1}{n!} \left( \int f(x) \, dx \right)^n
\]

[Hint: consider how the integral behaves under permutation of the \( x_i \)'s.]

3. Let \( f(x) \) be a non-negative Lebesgue measurable function on \( \mathbb{R} \), and let

\[
\phi(t) = m\{x : f(x) > t\}
\]

Show that \( \phi \) is right-continuous and decreasing, and that

\[
\int_0^\infty \phi(t) \, dt = \int f(x) \, dx
\]

4. 

(a.) If \( f, g \in L^1(\mathbb{R}, dx) \), show that \( f(x - y)g(y) \in L^1(\mathbb{R}, dx) \) for almost all \( x \).

(b.) If \( h(x) = \int f(x - y)g(y) \, dy \) (where defined), show that \( h \in L^1(\mathbb{R}, dx) \) and

\[
\int |h| \, dx \leq \left( \int |f| \, dx \right) \left( \int |g| \, dx \right)
\]