1. Let $\mu$ and $F$ be as in Theorem 1.16, Show that continuity of $F$ is equivalent to the statement $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$

2. Suppose that $\mu$ is a finite Borel measure on $\mathbb{R}$ with $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Show that, given $\epsilon > 0$, there exists $\delta > 0$, such that $\mu(E) < \epsilon$ if $\text{diam}(E) < \delta$.

3. Let $\mu$ be a finite Borel measure on $\mathbb{R}$, and $f_n(x)$ be a sequence of continuous functions on $\mathbb{R}$ such that $\lim_{n \to \infty} f_n(x) = 0$ for all $x$. Show that, given $\epsilon > 0$ and $\delta > 0$, there exists a set $E \subset \mathbb{R}$ with $\mu(E) < \delta$, and some $N < \infty$, such that $|f_n(x)| < \epsilon$ for $x \in \mathbb{R}\setminus E$ and $n \geq N$.

4. For a sequence $a \in \mathbb{Z}^N$ (these are called ternary sequences), let $N$ be the index where $a_N = 1$, but $a_n \in \{0, 2\}$ for $n < N$. Set $N = \infty$ if $a$ is a Cantor sequence. Define

$$\tilde{f}(a) = \frac{1}{2^N} + \frac{1}{2} \sum_{j=1}^{N-1} a_n \cdot 2^n.$$ 

with the understanding that $1/2^\infty = 0$ and $\infty - 1 = \infty$.

(a.) Suppose that $\sum_{n=1}^{\infty} \frac{a_n}{3^n} = \sum_{n=1}^{\infty} \frac{b_n}{3^n}$. Show that $\tilde{f}(a) = \tilde{f}(b)$. Thus, we can define $f(x)$ for $x \in [0, 1]$ by setting $f(x) = \tilde{f}(a)$ where $a$ is a ternary expansion of $x$. (See the first homework regarding potential non-uniqueness of ternary expansions.)

(b.) Show that $f : [0, 1] \to [0, 1]$ is monotonic increasing, and that the image of the Cantor set $C$ is the entire interval $[0, 1]$.

(c.) Show that $f$ is continuous, and that if $I$ is an open interval contained in $[0, 1]\setminus C$, then $f$ is constant on $I$.

(d.) Extend $f$ to a map $f : \mathbb{R} \to [0, 1]$ by setting $f(x) = 0$ if $x \leq 0$ and $f(x) = 1$ if $x \geq 1$.

Let $\mu$ be the finite Borel measure determined by $f$ as in Theorem 1.16. Show that $\mu(\mathbb{R}\setminus C) = 0$.
