Lecture 8: The Residue Theorem

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Residue Theorem

Suppose Γ is a cycle in E such that $\text{ind}_\Gamma(z) = 0$ for $z \notin E$. Suppose that f is analytic on $E \setminus \{z_1, \ldots, z_n\}$; that is, f has isolated singularities at a finite set of points in E. Then

$$\int_\Gamma f(w)\,dw \;=\; 2\pi i \sum_{j=1}^{n} \text{ind}_\Gamma(z_j) \cdot \text{Res}(f, z_j)$$

Proof. Take r small so $\overline{D_r}(z_j) \subset E$ contains only one singularity.

$$\text{Then :} \quad \Gamma' = \Gamma - \text{ind}_\Gamma(z_1)\partial D_r(z_1) - \cdots - \text{ind}_\Gamma(z_n)\partial D_r(z_n)$$

satisfies $\text{ind}_{\Gamma'}(z) = 0$ for all $z \notin E \setminus \{z_1, \ldots, z_n\}$.

By Cauchy's Theorem : $\displaystyle\int_{\Gamma'} f(w)\,dw \;=\; 0\,,$ so

$$\int_\Gamma f(w)\,dw = \sum_{j=1}^{n} \text{ind}_\Gamma(z_j) \int_{\partial D_r(z_0)} f(w)\,dw = 2\pi i \sum_{j=1}^{n} \text{ind}_\Gamma(z_j) \cdot \text{Res}(f, z_j)$$

Examples

In most cases of interest:

- Γ is (equivalent to) a closed path.
- $f(z)$ may have infinitely many singularities (like $\tan z$), but Γ encloses finitely many z_j, and $\mathrm{ind}_\Gamma(z_j) = \pm 1$ at each.

$$\int_{|z|=2\pi} \tan w \, dw \;=\; 2\pi i \sum_{k=-2}^{1} \mathrm{Res}\big(\tan z, (k+\tfrac{1}{2})\pi\big) \;=\; -8\pi i$$

$$\int_{|z|=3} \frac{w+3}{(w+1)(w-2)} \, dw \;=\; 2\pi i \left(\frac{5}{3} - \frac{2}{3}\right) \;=\; 2\pi i$$

Evaluate: $\displaystyle\int_\Gamma \frac{e^{iz}}{z^2+1}\,dz$, where $\Gamma = \gamma_1 + \gamma_2$:

$$\gamma_1 = [-R, R], \qquad \gamma_2(t) = Re^{it}, \;\; t \in [0, \pi], \quad \text{with } R > 1.$$

Simple poles at $z = \pm i$, $\quad \mathrm{ind}_\Gamma(i) = 1$, $\quad \mathrm{ind}_\Gamma(-i) = 0$.

$$\mathrm{Res}\left(\frac{e^{iz}}{z^2+1}, i\right) = \lim_{z\to i} \frac{(z-i)e^{iz}}{z^2+1} = \lim_{z\to i} \frac{e^{iz}}{z+i} = \frac{e^{-1}}{2i}$$

$$\int_\Gamma \frac{e^{iz}}{z^2+1}\,dz = 2\pi i\,\mathrm{Res}\left(\frac{e^{iz}}{z^2+1}, i\right) = \pi\, e^{-1}$$

If take $\gamma_2(t) = Re^{-it}$, $t \in [0, \pi]$, then

$$\int_\Gamma \frac{e^{iz}}{z^2+1}\,dz = -2\pi i\,\mathrm{Res}\left(\frac{e^{iz}}{z^2+1}, -i\right) = \pi\, e$$

Calculating residues for higher order poles

Pole of order 2 at z_0 : express the function $f(z)$ as $\dfrac{g(z)}{(z-z_0)^2}$,

$$g(z) = g(z_0) + g'(z_0)(z - z_0) + \cdots \Rightarrow \text{Res}\left(\frac{g(z)}{(z-z_0)^2}, z_0\right) = g'(z_0)$$

- $\dfrac{e^z}{(z-2)^2}$: $g(z) = e^z,\ g'(z) = e^z, \quad \text{Res}\left(\dfrac{e^z}{(z-2)^2}, 2\right) = e^2$

- $\dfrac{e^{iz}}{(z^2+1)^2} = \dfrac{e^{iz}}{(z-i)^2(z+i)^2}$ at $z_0 = i$: $g(z) = \dfrac{e^{iz}}{(z+i)^2}$

$$\text{Res}\left(\frac{e^{iz}}{(z^2+1)^2}, i\right) = \left.\frac{ie^{iz}}{(z+i)^2} - \frac{2e^{iz}}{(z+i)^3}\right|_{z=i} = \frac{-ie^{-1}}{2}$$

- $\dfrac{e^z}{(e^z-1)^2}$ at $z_0 = 0$.

Method: write $e^z - 1 = z\,h(z), \quad h(z) = 1 + \frac{1}{2!}z + \frac{1}{3!}z^2 + \cdots$

$$\frac{e^z}{(e^z-1)^2} = \frac{g(z)}{z^2} \quad \text{where} \quad g(z) = \frac{e^z}{h(z)^2}$$

$$g'(z) = \frac{e^z}{h(z)^2} - \frac{2e^z h'(z)}{h(z)^3}$$

Read off from Taylor expansion for h : $h(0) = 1, \; h'(0) = \frac{1}{2}$.

$$\operatorname{Res}\left(\frac{e^z}{(e^z-1)^2}, 0\right) = 0$$