Recall: if $z = x + iy$,  $w = u + iv$,

$$\text{Re}(\overline{w}z) = ux + vy = (u, v) \cdot (x, y)$$

- A line $L$ perpendicular to $w$ is given by:

$$\text{Re}(\overline{w}z) = c, \quad \text{where } c = \text{Re}(\overline{w}z_0) \text{ for any } z_0 \in L$$

Circle with center $z_0$, radius $r$:

$$|z - z_0|^2 = r^2,$$

$$|z|^2 - 2 \text{Re}(\overline{z}_0z) + |z_0|^2 = r^2$$

The general form of a circle:

$$|z|^2 - 2 \text{Re}(\overline{z}_0z) = c, \quad \text{where } c = r^2 - |z_0|^2$$
Theorem. Suppose $f$ is a linear fractional transformation. If $E$ is a circle or line in $\mathbb{C}$, then its image $f(E)$ is a circle, unless $E$ passes through the pole of $f$, in which case $f(E)$ is a line.

Proof. First check for $f(z) = 1/z$.

$E=$Line: given by $\text{Re}(\overline{w}z) = c$, image is $z : \text{Re}(\overline{w}z^{-1}) = c$,

Multiply by $z\overline{z} = |z|^2 : \text{Re}(wz) = c|z|^2$

Line if $c = 0$ (so $E$ passes through origin), circle if $c \neq 0$.

$E=$Circle: $|z|^2 - 2\text{Re}(\overline{z}_0 z) = c$, image is $z$ such that

$|z^{-1}|^2 - 2\text{Re}(\overline{z}_0 z^{-1}) = c$

Multiply by $z\overline{z} = |z|^2$ to turn this into

$1 - 2\text{Re}(z_0 z) = c|z|^2$

Line if $c = 0$ (so $E$ passes through origin), circle if $c \neq 0$. 
General case:

- Each LFT is either linear, or of the form: \( f(z) = L_1 \left( \frac{1}{L_2(z)} \right) \)
  for linear maps \( L_1, L_2 \).

- Linear maps take circles and lines to circles and lines.

Easiest way to determine image: use three points

**Example:** \( f(z) = \frac{z + 1}{z - 1} \)

- \( E = \{ z : |z - \frac{1}{2}| = \frac{1}{2} \} \supset \{ 0, 1, \frac{1+i}{2} \} \)

Image contains \( \{-1, \infty, -1-2i\} \), so it's the line \( \text{Re}(z) = -1 \)

- \( E = \{ z : \text{Re}(z) = 0 \} \supset \{ 0, i, -i \} \)

Image contains \( \{-1, -i, i\} \), so it's the circle \( |z|^2 = 1 \)