Linear Fractional Transformation (LFT): \( f(z) = \frac{az + b}{cz + d} \)

where \( a, b, c, d \in \mathbb{C} \), and \( ad - bc \neq 0 \).

- \( f(z) \) does not change if multiply \((a, b, c, d)\) by same number.
- If \( c = 0 \), then \( f(z) \) is linear. If \( c \neq 0 \), simple pole at \( z = -d/c \).
- We set \( f(\infty) = \lim_{z \to \infty} f(z) = a/c \), and \( f(\infty) = \infty \) if \( c = 0 \).
  
  We define \( f(-d/c) = \infty \) (so again \( f(\infty) = \infty \) if \( c = 0 \)).
  
  Then \( f(z) \) is now defined as a map \( f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} \).
- Fixed points: for \( z \in \mathbb{C} \), \( f(z) = z \iff az + b = cz^2 + dz \).
  
  If \( f \) has 3 or more fixed points (including \( \infty \)), then \( f(z) = z \).
Associate to matrix a LFT: \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \Rightarrow f(z) = \frac{az + b}{cz + d}
\]

\[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \cdot \begin{bmatrix}
a' & b' \\
c' & d'
\end{bmatrix} \Rightarrow f(g(z)) \quad \text{where} \quad g(z) = \frac{a'z + b'}{c'z + d'}
\]

In particular: \[f^{-1}(z) = \frac{dz - b}{-cz + a}\]

**Theorem**

Given two sets of 3 points \(\{z_0, z_1, z_2\}, \{w_0, w_1, w_2\} \subset \mathbb{C} \cup \{\infty\}\), there exists a unique LFT such that \(f(z_j) = w_j\) for \(j = 0, 1, 2\).

**Uniqueness**: \(f, g\) two such maps, then \(f \circ g^{-1}\) has 3 distinct fixed points, so \(f(g^{-1}(w)) = w\), hence \(f(z) = g(z)\).
Conformal automorphisms of $\mathbb{D} = \{z : |z| < 1\}$

If $|b| < 1$, consider the map $h_b$

$$h_b(z) = \frac{z - b}{1 - bz}$$

for which $h_b^{-1} = h_{-b}$.

- Pole of $h_b$ is at $z = 1/b \in \{z : |z| > 1\}$, so $h_b$ analytic on $\mathbb{D}$.

- If $z \in \partial \mathbb{D}$, so $z\bar{z} = 1$, then

$$|h_b(z)| = \left| \frac{1}{z} \frac{z - b}{\bar{z} - b} \right| = 1$$

- By the Maximum Modulus Theorem, $|h_b(z)| < 1$ if $|z| < 1$.

- Same holds for $h_b^{-1}$, so

**Fact**

$h_b$ is a 1-1, analytic map of $\mathbb{D}$ onto $\mathbb{D}$, $h_b(b) = 0$, $h_b(0) = -b$. 
Conformal automorphisms of $\mathbb{D} = \{ z : |z| < 1 \}$

**Theorem**

Every conformal equivalence from $\mathbb{D}$ to $\mathbb{D}$ must be of the form

$$f(z) = e^{i\theta} \frac{z - b}{1 - \bar{b}z} \quad \text{for some } \theta \in [0, 2\pi), \ b \in \mathbb{D}.$$

**Proof.** If $f : \mathbb{D} \to \mathbb{D}$ is 1–1, onto, and $f(b) = 0$, let $g = f \circ h_{-b}$.

$$g : \mathbb{D} \xrightarrow{1-1, \text{onto}} \mathbb{D} \quad \text{and} \quad g(0) = 0 \implies g(z) = e^{i\theta} z$$

for some $\theta \in [0, 2\pi)$ by Theorem 3.5.6. Then

$$f(z) = g(h_b(z)) = e^{i\theta} h_b(z).$$