Lecture 22: Conformal Mappings

Hart Smith

Department of Mathematics
University of Washington, Seattle

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Conformal mappings on the plane

Consider a differentiable mapping \( f(x, y) = (u(x, y), v(x, y)) \), and the linearization of \( f \) at a point \((x, y)\):

\[
\begin{bmatrix}
    u(x + \Delta x, y + \Delta y) \\
    v(x + \Delta x, y + \Delta y)
\end{bmatrix} \approx \begin{bmatrix}
    u(x, y) \\
    v(x, y)
\end{bmatrix} + Df(x, y) \cdot \begin{bmatrix}
    \Delta x \\
    \Delta y
\end{bmatrix}
\]

where \( Df(x, y) = \begin{bmatrix}
    u_x(x, y) & u_y(x, y) \\
    v_x(x, y) & v_y(x, y)
\end{bmatrix} \)

Geometric Definition of conformal

The map \( f \) is conformal if, at each \((x, y)\), the matrix \( Df(x, y) \) is non-singular and angle-preserving. Equivalently,

\[
Df(x, y) = \begin{bmatrix}
    r \cos \theta & -r \sin \theta \\
    r \sin \theta & r \cos \theta
\end{bmatrix} = \begin{bmatrix}
    r & 0 \\
    0 & r
\end{bmatrix} \cdot \begin{bmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{bmatrix}
\]

where \( r \) and \( \theta \) depend on \((x, y)\), and \( r \neq 0 \).
Lemma

\[ f(x, y) = (u(x, y), v(x, y)) \] is conformal if and only if the C-R equations, \( u_x = v_y \) and \( u_y = -v_x \) hold, and \( \text{D}f \neq 0 \).

Proof. If 
\[
\begin{bmatrix}
u_x & u_y \\
v_x & v_y
\end{bmatrix}
= 
\begin{bmatrix}
r \cos \theta & -r \sin \theta \\
r \sin \theta & r \cos \theta
\end{bmatrix}
\] then C-R eqn’s hold.

Conversely, C-R eqn’s say 
\[
\begin{bmatrix}
u_x & u_y \\
v_x & v_y
\end{bmatrix}
\] takes the form 
\[
\begin{bmatrix}
a & -b \\
b & a
\end{bmatrix}.
\]

The point \((a, b)\) lies on the circle of radius \(r = \sqrt{a^2 + b^2}\), so we can write \(a = r \cos \theta, \ b = r \sin \theta\), for some \(\theta\).

Corollary

An analytic function \(f(z)\) is conformal at points where \(f'(z) \neq 0\), where we identify the complex numbers \(\mathbb{C}\) with the plane \(\mathbb{R}^2\).

Remark. If write \(f'(z) = re^{i\theta}\), then get the same \(r, \theta\) above.
Grid representation of conformal map $z \rightarrow e^z$
Grid representation of conformal map $z \rightarrow z^{\frac{1}{2}}$
Conformal equivalence

**Definition**
We say two open sets $U$ and $V$ in $\mathbb{C}$ are conformally equivalent if there is an analytic map $f : U \to V$ that is 1-1 and onto. Such an $f$ is called a conformal equivalence between $U$ and $V$.

- $f^{-1}(w)$ is then a conformal equivalence between $V$ and $U$.

**Examples.**
- $f(z) = e^z$ is a conformal equivalence between
  $$U = \{ z : -\pi < \text{Im}(z) < \pi \} \quad \text{and} \quad V = \mathbb{C} \setminus (-\infty, 0]$$
- $f(z) = z^{\frac{1}{2}}$ is a conformal equivalence between
  $$U = \{ z : \text{Im}(z) > 0 \} \quad \text{and} \quad V = \{ w : \text{Im}(w) > 0 \text{ and } \text{Re}(w) > 0 \}$$
\{ z : \text{Re}(z) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \} \rightarrow \mathbb{C} \setminus (-i\infty, -i] \cup [i, i\infty)