Lecture 21: More Fourier transforms. The Open Mapping Theorem

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Theorem

Assume $P(z)$, $Q(z)$ polynomials, and $\text{order}(Q) \geq \text{order}(P) + 1$, and $Q(z)$ has no zeroes on the real number line.

If $s < 0$:
\[
\int_{-\infty}^{\infty} e^{-ist} \frac{P(t)}{Q(t)} \, dt = 2\pi i \sum_{z_j} \text{Res} \left( e^{-isz} \frac{P(z)}{Q(z)}, z_j \right)
\]
with $\{z_j\}$ the zeroes of $Q$ in upper half plane $\text{Im}(z) > 0$.

If $s > 0$:
\[
\int_{-\infty}^{\infty} e^{-ist} \frac{P(t)}{Q(t)} \, dt = -2\pi i \sum_{w_k} \text{Res} \left( e^{-isz} \frac{P(z)}{Q(z)}, w_k \right)
\]
where $\{w_k\}$ are the zeroes of $Q$ in lower half plane $\text{Im}(w) < 0$.

If $s = 0$: integral converges only if $\text{order}(Q) \geq \text{order}(P) + 2$. 
Evaluate: \( \hat{f}(s) \) for \( s \neq 0 \), where \( f(t) = \frac{1}{1 - i t} \).

For \( s < 0 \):
\[
\hat{f}(s) = \lim_{R \to \infty} \int_{[-R,R]+\mu_R^+} \frac{e^{-isz}}{1 - iz} \, dz
\]
\[
= 0 \quad \text{(no poles)}
\]

For \( s > 0 \):
\[
\hat{f}(s) = \lim_{R \to \infty} \int_{[-R,R]+\mu_R^-} \frac{e^{-isz}}{1 - iz} \, dz
\]
\[
= -2\pi i \text{Res}\left(\frac{e^{-isz}}{1 - iz}, -i\right) = 2\pi e^{-s}
\]

For \( s = 0 \):
\[
\int_{-\infty}^{\infty} \frac{1}{1 - it} \, dt \quad \text{depends on how you take limits.}
\]

Example: for symmetric limits
\[
\lim_{R \to \infty} \int_{-R}^{R} \frac{1}{1 - it} \, dt = \pi
\]
Evaluate: \( \hat{f}(s) \) for \( s \neq \pm 1 \), where \( f(t) = \frac{\sin t}{t} \).

\[
\hat{f}(s) = \lim_{R \to \infty} \int_{[-R,R]} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} \, dz
\]

\( s > 1 \) : \( \hat{f}(s) = \lim_{R \to \infty} \int_{[-R,R] + \mu_R^{-}} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} \, dz = 0 \)

\( s < -1 \) : \( \hat{f}(s) = \lim_{R \to \infty} \int_{[-R,R] + \mu_R^{+}} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} \, dz = 0 \)

\( -1 < s < 1 \) : “go around 0” to handle exponentials separately.

\[
\lim_{R \to \infty} \int_{[-R,R]} \frac{e^{-i(s-1)z} - e^{-i(s+1)z}}{2iz} \, dz = 2\pi i \text{Res} \left( \frac{e^{-i(s-1)z}}{2iz}, 0 \right) = \pi
\]
Theorem: assume $f$ analytic on open set $E$

Suppose that $f^{(k)}(z_0) = 0$ for $1 \leq k < m$, and $f^{(m)}(z_0) \neq 0$. Let $f(z_0) = w_0$. Then for some $r, \delta > 0$, if $w \in D_\delta(w_0) \setminus \{w_0\}$, the equation $f(z) = w$ has $m$ distinct solutions for $z \in D_r(z_0)$.

- Assumptions say $f(z) - w_0$ has a zero of order $m$ at $z = z_0$.
- Zeroes of $f(z) - w_0$ and of $f'(z)$ are isolated, so exists $r > 0$:

$$
\min_{|z - z_0| = r} |f(z) - w_0| = \delta > 0, \quad f'(z) \neq 0 \text{ if } 0 < |z - z_0| < r
$$

- If $|w - w_0| < \delta$, then

$$
|(f(z) - w) - (f(z) - w_0)| \leq |f(z) - w_0| \text{ for } z \in \partial D_r(z_0).
$$

By Rouché, $f(z) - w$ has $m$ zeroes for $z \in D_r(z_0)$, which are simple if $w \neq w_0$ since $f'(z) \neq 0$ if $0 < |z - z_0| < r$. 
Open Mapping Theorem: assume $U$ is open, connected

If $f(z)$ is analytic on $U$ and not constant, then $f(U) \subset \mathbb{C}$ is open.

- “$f$ maps open sets to open sets, unless $f$ is constant”.

Proof. By Theorem, if $w_0 \in f(U)$, $\exists \delta > 0$ with $D_\delta(w_0) \subset f(U)$. 