Lecture 2: Chains and Cycles

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Some definitions from 427

- path $=$ continuous, piecewise smooth map $\gamma : [a, b] \to \mathbb{C}$.
- closed path $=$ path such that $\gamma(b) = \gamma(a)$.

Definition

If f is a continuous function on $E \subset \mathbb{C}$, and γ is a path in E, the contour integral of f over γ is

$$\int_\gamma f(w)\, dw = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt$$

By Fundamental Theorem of Calculus: if f is analytic on E

$$\int_\gamma f'(w)\, dw = f(\gamma(b)) - f(\gamma(a))$$

Two theorems from 427

Cauchy's theorem for convex sets

Suppose that f is analytic on an open convex set E.
If γ is a closed path in E, then $\displaystyle\int_\gamma f(w)\,dw \;=\; 0\,.$

Cauchy integral formula for an open, convex set E

If f is analytic on E, γ a closed path in E, then for $z \in E \setminus \{\gamma\}$:

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw \;=\; \operatorname{ind}_\gamma(z)\cdot f(z)$$

Here $\operatorname{ind}_\gamma(z)$, defined for $z \in \mathbb{C} \setminus \{\gamma\}$, is an integer, given by

$$\operatorname{ind}_\gamma(z) \;=\; \frac{1}{2\pi i}\int_\gamma \frac{1}{w-z}\,dw$$

Theorem

Assume $\gamma : [a, b] \to \mathbb{C}$ is a path (not necessarily closed), and f is continuous on $\{\gamma\}$. Then the function $g(z) = \int_\gamma \frac{f(w)}{w - z}\, dw$ is analytic on $\mathbb{C} \setminus \{\gamma\}$, and $g'(z) = \int_\gamma \frac{f(w)}{(w - z)^2}\, dw$.

Proof. Assume $z \notin \{\gamma\}$, and $z + h \notin \{\gamma\}$.

$$\frac{g(z+h) - g(z)}{h} = \frac{1}{h} \int_\gamma f(w) \Big(\frac{1}{w - z - h} - \frac{1}{w - z} \Big)\, dw$$

$$= \int_\gamma \frac{f(w)}{(w - z - h)(w - z)}\, dw$$

$$\lim_{h \to 0} \frac{f(w)}{(w - z - h)(w - z)} = \frac{f(w)}{(w - z)^2}$$ uniformly over $w \in \{\gamma\}$,

so the difference quotient converges to $\int_\gamma \frac{f(w)}{(w - z)^2}\, dw$.

Chains

- A *chain* is a finite collection of paths: $\Gamma = \{\gamma_1, \dots, \gamma_n\}$.
- We define $\displaystyle\int_\Gamma f(w)\,dw = \sum_{j=1}^{n} \int_{\gamma_j} f(w)\,dw$
- Given a set E and two chains Γ, Γ' contained in E, we say Γ' is *equivalent* to Γ if for all continuous functions f on E,

$$\int_\Gamma f(w)\,dw = \int_{\Gamma'} f(w)\,dw$$

Example. $\{\gamma_1, \dots, \gamma_n\}$ is equivalent to $\{\gamma_1, \dots, \gamma_n, \gamma, -\gamma\}$ for any path γ in E, where $-\gamma$ is the path γ going backwards, since $\displaystyle\int_{-\gamma} f(w)\,dw = -\int_\gamma f(w)\,dw$ for all f.

Cycles

- Given two paths $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ and $\gamma_2 : [a_2, b_2] \to \mathbb{C}$, if $\gamma_1(b_1) = \gamma_2(a_2)$, we can join them to form a path $\gamma_1 \cup \gamma_2$:

$$\gamma_1 \cup \gamma_2(t) = \begin{cases} \gamma_1(t), & a_1 \le t \le b_1 \,, \\ \gamma_2(t + a_2 - b_1), & b_1 \le t \le b_1 + (b_2 - a_2)\,. \end{cases}$$

- $\{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_n\}$ is equivalent to $\{\gamma_1 \cup \gamma_2, \gamma_3, \ldots, \gamma_n\}$
- We say a chain Γ is a *cycle* if we can turn Γ into a collection of closed paths by successively joining together paths.

Example. More than one way of seeing the following is a cycle:

$$\{\, [0,1]\,,\, [1,i]\,,\, [i,0]\,,\, [0,-i]\,,\, [-i,-1]\,,\, [-1,0]\,\}$$