Assume that: $f(z)$ is analytic on $D_1(0) = \{ z : |z| < 1 \}$, and continuous on $\overline{D_1}(0) = \{ z : |z| \leq 1 \}$.

By the Maximum Modulus Theorem:

- If $|f(z)| \leq 1$ when $|z| = 1$, then $|f(z)| \leq 1$ when $|z| \leq 1$,
- $|f(z)| < 1$ when $|z| < 1$ unless $f(z)$ is constant.

Example: the function $f(z) = \frac{z + \frac{1}{2}}{\frac{1}{2}z + 1}$

If $|z| = 1$, then $z\overline{z} = 1$, so:

$$|f(z)| = \left| \frac{z + \frac{1}{2}}{\frac{1}{2}z + z\overline{z}} \right| = \frac{1}{|z|} \left| \frac{z + \frac{1}{2}}{\frac{1}{2} + \overline{z}} \right| = 1$$

Therefore: $|f(z)| < 1$ if $|z| < 1$, $|f(z)| = 1$ if $|z| = 1$. 
Theorem: assume $f$ analytic on $D_1(0)$, continuous on $\overline{D_1(0)}$.

Suppose that $|f(z)| = 1$ when $|z| = 1$. If $f(z)$ is not constant, then there is some point $z \in D_1(0)$ where $f(z) = 0$.

**Proof.** By Maximum Modulus, $|f(z)| < 1$ when $|z| < 1$.

- If $f(z) \neq 0$ on $D_1(0)$, then $1/f(z)$ is analytic, continuous.
- By assumption, $|1/f(z)| = 1/|f(z)| = 1$ if $|z| = 1$.
- Max Mod implies $1/|f(z)| < 1$ if $|z| < 1$, a contradiction.

**Stronger fact:** if $|w| < 1$, then $w = f(z)$ for some $|z| < 1$.

Typical such map: $f(z) = z^n$, $n \geq 1$. 
Schwarz’s Lemma

Assume that \( f(z) \) is analytic on \( D_1(0) \), and \(|f(z)| \leq 1\) for \(|z| < 1\).

If \( f(0) = 0 \), then \(|f(z)| \leq |z|\) for all \(|z| < 1\), and \(|f'(0)| \leq 1\).

If \(|f'(0)| = 1\), or \(|f(z)| = |z|\) some \( z \), then \( f(z) = cz, \ |c| = 1\).

**Proof.** The function \( g(z) = \begin{cases} \frac{f(z)}{z}, & 0 < |z| < 1, \\ f'(0), & z = 0, \end{cases} \) is analytic on \( D_1(0) \). For every \( r < 1 \):

\[
\text{if } |z| = r : \quad |g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}.
\]

By Max Mod: \(|g(z)| \leq r^{-1}\) if \(|z| < r\). This holds for all \( r < 1 \), so

\[
|g(z)| \leq 1 \quad \text{if} \quad |z| < 1.
\]

If \(|g(z)| = 1\) for some \(|z| < 1\), i.e. \(|f'(0)| = 1\) or \(|f(z)| = |z|\),

by Max Mod \( g(z) = c \), so \( f(z) = cz \).
Theorem

Assume $f(z)$ is a 1-1 map of $D_1(0)$ onto $D_1(0)$, and $f$ and $f^{-1}$ are analytic functions. If $f(0) = 0$, then $f(z) = cz$, with $|c| = 1$.

Proof. Schwarz’s lemma applies to both $f(z)$ and $f^{-1}(z)$:

- $f(0) = 0$ so $|f'(0)| \leq 1$, and $f^{-1}(0) = 0$ so $|(f^{-1})'(0)| \leq 1$.
- Differentiate $f(f^{-1}(z)) = z$: by chain rule $f'(0)(f^{-1})'(0) = 1$.
- Conclude $|f'(0)| = 1$, so $f(z) = cz$ with $|c| = 1$.

Result fails if $f(z)$ is not 1-1: example $f(z) = z^2$. 