Lecture 16: Calculating Residues

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Easiest case

$$f(z) = \frac{g(z)}{(z-z_0)^k}, \quad g \text{ analytic on } D_r(z_0)$$

$$\text{Res}(f, z_0) \; = \; \frac{g^{(k-1)}(z_0)}{(k-1)!}$$

Proof. Taylor expansion $g(z) = \sum_{j=0}^{\infty} \frac{1}{j!} g^{(j)}(z_0)\,(z-z_0)^j$

$$\frac{g(z)}{(z-z_0)^k} \; = \; \ldots + \frac{g^{(k-1)}(z_0)}{(k-1)!} \frac{1}{(z-z_0)} + \ldots$$

Special cases: $k = 1, 2,$

$$\text{Res}\left(\frac{g(z)}{(z-z_0)}, z_0\right) = g(z_0)\,, \quad \text{Res}\left(\frac{g(z)}{(z-z_0)^2}, z_0\right) = g'(z_0)\,.$$

Examples with $k = 1, 2$.

- $$f(z) = \frac{\log(z)}{(z-5)^2}, \qquad \mathrm{Res}(f,5) = (\log)'(5) = \frac{1}{5}$$

- $$f(z) = \frac{\log(z)}{z^2-5} = \frac{\log(z)}{(z-\sqrt{5})(z+\sqrt{5})},$$

  $$\mathrm{Res}(f,\sqrt{5}) = \frac{\log(\sqrt{5})}{2\sqrt{5}}$$

- $$f(z) = \frac{\cos z}{(z^2+4)^2} = \frac{\cos(z)}{(z-2i)^2(z+2i)^2},$$

  $$\mathrm{Res}(f,2i) = \left(\frac{\cos(z)}{(z+2i)^2}\right)'\bigg|_{z=2i} = \frac{-\sin(2i)}{(4i)^2} - \frac{2\cos(2i)}{(4i)^3}$$

Long division

$$\text{Res}\left(\frac{e^z}{1-\cos z},0\right) = \text{Res}\left(\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots}{\frac{1}{2}z^2-\frac{1}{24}z^4+\cdots},0\right)$$

$$= \text{Res}\left(\frac{2}{z^2}\cdot\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\cdots}{1-\frac{1}{12}z^2+\cdots},0\right)$$

Use long division of first few terms... need coefficient of z

$$1-\tfrac{1}{12}z^2 \;\overline{\big)\; 1+z+\tfrac{1}{2}z^2+\tfrac{1}{6}z^3+\cdots}$$

Answer to long division: $1+z+\frac{7}{12}z^2+\cdots$

$$\text{Res}\left(\frac{e^z}{1-\cos z},0\right) = \text{Res}\left(\frac{2}{z^2}\cdot\left(1+z+\frac{7}{12}z^2+\cdots\right),0\right) = 2$$

- Needed include only to order 1 in numerator, denominator.

Long division

$$\operatorname{Res}\left(\frac{e^z}{e^z - z - 1}, 0\right) = \operatorname{Res}\left(\frac{1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots}{\frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots}, 0\right)$$

$$= \operatorname{Res}\left(\frac{2}{z^2} \cdot \frac{1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \cdots}{1 + \frac{1}{3}z + \cdots}, 0\right)$$

$$= \operatorname{Res}\left(\frac{2}{z^2} \cdot \frac{1 + z}{1 + \frac{1}{3}z}, 0\right)$$

Long division

$$1 + \tfrac{1}{3}z \;\big)\overline{\,1 + z\,} \;=\; 1 + \tfrac{2}{3}z + \cdots$$

$$\operatorname{Res}\left(\frac{e^z}{e^z - z - 1}, 0\right) = \operatorname{Res}\left(\frac{2}{z^2} \cdot \left(1 + \tfrac{2}{3}z + \cdots\right), 0\right) = \frac{4}{3}$$

Example of third order pole

$$\operatorname{Res}\left(\frac{\cos z}{z^2 \sin z}, 0\right) = \operatorname{Res}\left(\frac{1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \cdots}{z^2(z - \frac{1}{6}z^3 + \cdots)}, 0\right)$$

$$= \operatorname{Res}\left(\frac{1}{z^3} \cdot \frac{1 - \frac{1}{2}z^2 + \frac{1}{24}z^4 + \cdots}{1 - \frac{1}{6}z^2 + \cdots}, 0\right)$$

$$= \operatorname{Res}\left(\frac{1}{z^3} \cdot \frac{1 - \frac{1}{2}z^2}{1 - \frac{1}{6}z^2}, 0\right)$$

Long division

$$1 - \tfrac{1}{6}z^2 \overline{\big)\, 1 - \tfrac{1}{2}z^2} \;=\; 1 - \tfrac{1}{3}z^2 + \cdots$$

$$\operatorname{Res}\left(\frac{\cos z}{z^2 \sin z}, 0\right) = \operatorname{Res}\left(\frac{1}{z^3} \cdot (1 - \tfrac{1}{3}z^2 + \cdots), 0\right) = -\frac{1}{3}$$