Lecture 12: The Inverse Function Theorem

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Rouche’s Theorem

If $f, g$ are analytic on $E$, $\gamma$ a simple path in $E$ with $\text{int}(\gamma) \subset E$, $f, g$ have no zeroes on $\gamma$, and $\left| \frac{f(z)}{g(z)} - 1 \right| \leq 1$ for all $z \in \{\gamma\}$, then:

$$\#\{\text{zeroes of } f \text{ in } \gamma\} = \#\{\text{zeroes of } g \text{ in } \gamma\}.$$  

Another way to state criterion on $f$ and $g$:

$$|f(z) - g(z)| \leq |g(z)| \quad \text{for all} \quad z \in \{\gamma\}.$$
Assume \( f(z_0) = w_0 \), and \( f'(z_0) \neq 0 \). Then there exists \( f^{-1}(w) \):

- \( f(f^{-1}(w)) = w \) for \( w \in D_\delta(w_0) \), some \( \delta > 0 \).
- \( f^{-1}(w_0) = z_0 \); \( f^{-1}(w) \) is analytic on \( D_\delta(w_0) \).
- \( f^{-1}(w)' = 1/f'(f^{-1}(w)) \).

Existence of \( f^{-1}(w) \):

- The function \( f(z) - w_0 \) has a zero of order 1 at \( z = z_0 \).
- Zeroes are isolated: \( f(z) - w_0 \) has unique zero in \( \overline{D}_r(z_0) \) for some \( r > 0 \), so
  \[
  \min_{|z-z_0|=r} |f(z) - w_0| = \delta > 0.
  \]
- If \( |w - w_0| < \delta \), then
  \[
  |(f(z) - w) - (f(z) - w_0)| \leq |f(z) - w_0| \quad \text{for } z \in \partial D_r(z_0)
  \]
  so \( f(z) - w \) has a unique zero in \( D_r(z_0) \), which gives \( f^{-1}(w) \).
Let $f(z) = \tan z$, $z_0 = 0$, $w_0 = 0$. Then $\tan'(z_0) = 1$.

$\partial D_1(0) = \{ e^{it} : t \in [0, 2\pi] \}$

Image of $\partial D_1(0)$ under $\tan z$: $\{ \tan(e^{it}) : t \in [0, 2\pi] \}$
Analyticity of \( f^{-1}(w) \) for \(|w - w_0| < \delta\)

**Key fact:** \( f^{-1}(w) \) is unique zero of \( f(z) - w \) for \( z \in D_r(z_0) \),

\[
\text{Res}\left( \frac{f'(z)}{f(z) - w}, f^{-1}(w) \right) = 1
\]

\[\Rightarrow \frac{f'(z)}{f(z) - w} = \frac{1}{z - f^{-1}(w)} + \text{analytic}\]

**Explicit formula:**

\[
f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{z f'(z)}{f(z) - w} \, dz
\]

This is analytic on \(|w - w_0| < \delta\), and equals the power series

\[
f^{-1}(w) = \sum_{k=0}^{\infty} a_k (w - w_0)^k, \quad a_k = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{z f'(z)}{(f(z) - w_0)^{k+1}} \, dz
\]
Example

Let \( f(z) = \sin z \). Then \( f'(z) = \cos z \neq 0 \) if \( z \neq (k + \frac{1}{2})\pi \).

**Note:** \( \sin z = \pm 1 \Rightarrow \sin'z = 0 \); no analytic inverse at \( w_0 = \pm 1 \).

For a local inverse, two values are possible for \( (\sin^{-1})'(w) \):

\[
(\sin^{-1})'(w) = \frac{1}{\cos(\sin^{-1}(w))} = \frac{1}{\sqrt{1 - w^2}}
\]

Take \( z_0 = 0, \ w_0 = \sin(0) = 0 \). Then for \( |w| < \delta \):
there is a choice of \( \sin^{-1}(w) \) with \( \sin^{-1}(0) = 0 \), and

\[
(\sin^{-1})'(0) = \cos(\sin^{-1}(0)) = \cos(0) = 1.
\]

Take \( z_0 = \pi, \ w_0 = \sin(\pi) = 0 \). Then for \( |w| < \delta \):
there is a choice of \( \sin^{-1}(w) \) with \( \sin^{-1}(0) = \pi \), and

\[
(\sin^{-1})'(0) = \cos(\sin^{-1}(0)) = \cos(\pi) = -1.
\]