Lecture 12: The Inverse Function Theorem

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Rouche's Theorem

If f, g are analytic on E, γ a simple path in E with $\text{int}(\gamma) \subset E$,

f, g have no zeroes on γ, and $\left|\dfrac{f(z)}{g(z)} - 1\right| \leq 1$ for all $z \in \{\gamma\}$,

then: $$\#\big\{\text{zeroes of } f \text{ in } \gamma\big\} = \#\big\{\text{zeroes of } g \text{ in } \gamma\big\}.$$

Another way to state criterion on f and g:

$$|f(z) - g(z)| \leq |g(z)| \quad \text{for all} \quad z \in \{\gamma\}.$$

Assume $f(z_0) = w_0$, and $f'(z_0) \neq 0$. Then there exists $f^{-1}(w)$:

- $f\big(f^{-1}(w)\big) = w$ for $w \in D_\delta(w_0)$, some $\delta > 0$.
- $f^{-1}(w_0) = z_0$; $f^{-1}(w)$ is analytic on $D_\delta(w_0)$.
- $f^{-1}(w)' = 1/f'\big(f^{-1}(w)\big)$.

Existence of $f^{-1}(w)$:

- The function $f(z) - w_0$ has a zero of order 1 at $z = z_0$.
- Zeroes are isolated: $f(z) - w_0$ has unique zero in $\overline{D_r}(z_0)$ for some $r > 0$, so

$$\min_{|z-z_0|=r} |f(z) - w_0| = \delta > 0.$$

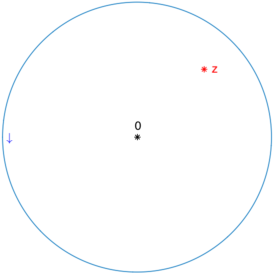
- If $|w - w_0| < \delta$, then

$$\Big|\big(f(z) - w\big) - \big(f(z) - w_0\big)\Big| \leq |f(z) - w_0| \text{ for } z \in \partial D_r(z_0)$$

so $f(z) - w$ has a unique zero in $D_r(z_0)$, which gives $f^{-1}(w)$.

Pictorial illustration behind existence of $f^{-1}(w)$

Let $f(z) = \tan z,\ z_0 = 0,\ w_0 = 0.$ Then $\tan'(z_0) = 1.$



$\partial D_1(0) = \{e^{it} : t \in [0, 2\pi]\}$

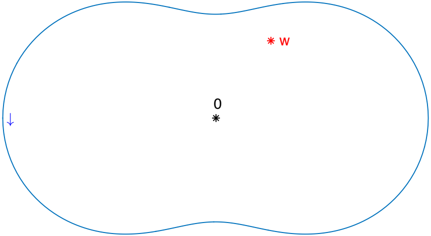


Image of $\partial D_1(0)$ under $\tan z$:

$\{\tan(e^{it}) : t \in [0, 2\pi]\}$

Analyticity of $f^{-1}(w)$ for $|w - w_0| < \delta$

Key fact: $f^{-1}(w)$ is unique zero of $f(z) - w$ for $z \in D_r(z_0)$,

$$\begin{aligned} \operatorname{Res}\left(\frac{f'(z)}{f(z)-w}, f^{-1}(w)\right) &= 1 \\ \Rightarrow \quad \frac{f'(z)}{f(z)-w} &= \frac{1}{z - f^{-1}(w)} + \text{analytic} \end{aligned}$$

Explicit formula : $$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{z\, f'(z)}{f(z) - w}\, dz$$

This is analytic on $|w - w_0| < \delta$, and equals the power series

$$f^{-1}(w) = \sum_{k=0}^{\infty} a_k (w - w_0)^k\,, \qquad a_k = \frac{1}{2\pi i} \int_{\partial D_r(z_0)} \frac{z\, f'(z)}{(f(z) - w_0)^{k+1}}\, dz$$

Example

Let $f(z) = \sin z$. Then $f'(z) = \cos z \neq 0$ if $z \neq (k + \frac{1}{2})\pi$.

Note: $\sin z = \pm 1 \Rightarrow \sin' z = 0$; no analytic inverse at $w_0 = \pm 1$.

For a local inverse, two values are possible for $(\sin^{-1})'(w)$:

$$(\sin^{-1})'(w) = \frac{1}{\cos\left(\sin^{-1}(w)\right)} = \frac{1}{\sqrt{1 - w^2}}$$

Take $z_0 = 0$, $w_0 = \sin(0) = 0$. Then for $|w| < \delta$:
there is a choice of $\sin^{-1}(w)$ with $\sin^{-1}(0) = 0$, and

$$(\sin^{-1})'(0) = \cos\left(\sin^{-1}(0)\right) = \cos(0) = 1.$$

Take $z_0 = \pi$, $w_0 = \sin(\pi) = 0$. Then for $|w| < \delta$:
there is a choice of $\sin^{-1}(w)$ with $\sin^{-1}(0) = \pi$, and

$$(\sin^{-1})'(0) = \cos\left(\sin^{-1}(0)\right) = \cos(\pi) = -1.$$