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Text: Complex Variables, Joseph Taylor (AMS, 2011)

Office Hours: Padelford C-447, MWF 2:00 - 3:30 pm.

Grading:

- Midterm, Wednesday February 13: 30%
- Final Exam, Wednesday March 20: 50%
- Weekly Homework: 20%
Theorem \( \mathbb{R} \)

Assume that \( f(t) \) is a continuous, real valued function on \([a, b]\), and let \( M = \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \). Then if \( f(t) \leq M \) for all \( t \in [a, b] \), we must have \( f(t) = M \) for all \( t \in [a, b] \).

Theorem \( \mathbb{C} \)

Assume \( f(t) \) is a continuous, complex valued function on \([a, b]\), and let \( c = \frac{1}{b-a} \int_{a}^{b} f(t) \, dt \). Then if \( |f(t)| \leq |c| \) for all \( t \in [a, b] \), we must have \( f(t) = c \) for all \( t \in [a, b] \).

Proof. Write \( c^{-1}f(t) = g(t) + ih(t) \), so \( |c^{-1}f(t)| \leq 1 \ \forall t \in [a, b] \).

Given: \( 1 = \frac{1}{b-a} \int_{a}^{b} g(t) + ih(t) \, dt \), so \( g(t) = 1 \) for all \( t \).
**Theorem:** suppose $f(z)$ is analytic on $D_r(z_0)$

If $|f(z)| \leq |f(z_0)|$ for all $z \in D_r(z_0)$, then $f(z) = f(z_0)$ on $D_r(z_0)$.

**Proof.** For any $r' < r$, by the Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_{|w - z_0| = r'} \frac{f(w)}{w - z_0} \, dw = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + r'e^{it}) \, dt$$

$$|f(z_0 + r'e^{it})| \leq |f(z_0)| \Rightarrow f(z_0 + r'e^{it}) = f(z_0) \text{ by Theorem } \mathbb{C}.$$  

This is true for any $r' < r$, so $f(z) = f(z_0)$ for all $z \in D_r(z_0)$.

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**Maximum modulus theorem**

Assume $f(z)$ is analytic on $E$, and continuous on $\bar{E}$, where $E$ is a bounded, connected, open set. Then the maximum of $|f(z)|$ on $\bar{E}$ occurs on $\partial E$ (and only on $\partial E$ if $f$ is not constant).

**Proof.** $|f(z)|$ can’t equal $\max_{\bar{E}} |f|$ for $z \in E$ unless $f$ is constant.
Examples

1. \(|z^4 - 1|\) on \(\{z : |z| \leq 1\}\). Maximum is at some \(z = e^{i\theta}\),

\[
|e^{4i\theta} - 1|^2 = (\cos(4\theta) - 1)^2 + \sin(4\theta)^2
\]

Setting the derivative equal to 0 gives \(\sin(4\theta) = 0\):

\[
\theta = 0, \pm \frac{\pi}{2}, \pi \text{ (minimum)}, \quad \theta = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4} \text{ (maximum)}
\]

2. \(|e^z|\) on the square: \(-1 \leq \text{Re}(z) \leq 1\), \(-1 \leq \text{Im}(z) \leq 1\).

\[
|e^{x+iy}| = e^x \quad \text{does not depend on } y.
\]

\(e^x\) is maximized over \(-1 \leq x \leq 1\) at \(x = 1\), so maximum is achieved at every point on right edge of the square.
Harmonic functions

**Definition**
A function $u(x, y)$ on an open set $E \subset \mathbb{R}^2$ is harmonic if:

$$\partial_x^2 u(x, y) + \partial_y^2 u(x, y) = 0 \quad \text{for all } (x, y) \in E.$$

**Key fact:** If $f = u + iv$ is analytic, then $u$ and $v$ are harmonic,

_The real and imaginary parts of an analytic function are harmonic._

The proof is an easy consequence of the

**Cauchy-Riemann equations**

$$\partial_x u(x, y) = \partial_y v(x, y), \quad \partial_y u(x, y) = -\partial_x v(x, y).$$
Theorem: assume $E \subset \mathbb{C}$ is open, convex

If $u$ is a real-valued, harmonic function on $E$, then there is a real-valued, harmonic function $v$ on $E$ so that $u + iv$ is analytic.

**Proof.** The function $g = \partial_x u - i \partial_y u$ is analytic on $E$, by the Cauchy-Riemann equations. Its anti-derivative $f(z)$ is analytic:

$$f(z) = u(x_0, y_0) + \int_{z_0}^{z} g(z) \, dz, \quad z_0 = x_0 + iy_0 \in E.$$  

The real part of $f(z)$ equals $u(x, y)$, since

$$f(z) = u(x_0, y_0) + \int_{z_0}^{z} \left( \partial_x u \, dx + \partial_y u \, dy \right)$$

$$+ i \int_{z_0}^{z} \left( \partial_x u \, dy - \partial_y u \, dx \right)$$

Choose $v = \int_{z_0}^{z} \left( \partial_x u \, dy - \partial_y u \, dx \right)$
Mean value property for harmonic functions

**Theorem**

Suppose $u$ is harmonic on $E$. Then, whenever $\overline{D_r(z_0)} \subset E$,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta$$

**Proof.** There is analytic $f$ on $\overline{D_r(z_0)} \subset E$, with $u(z) = \text{Re}(f(z))$,

$$u(z_0) = \text{Re}\left( \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta \right) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta$$

Applying our first Theorem to $u(z)$ gives us

**Maximum principle for harmonic functions**

Assume $u$ is harmonic on $E$, and continuous on $\overline{E}$, where $E$ is a bounded, connected, open set. Then the maximum of $u$ on $\overline{E}$ occurs on $\partial E$ (and only on $\partial E$ if $u$ is not constant).