Lecture 8: Branches of multi-valued functions

Hart Smith

Department of Mathematics
University of Washington, Seattle

Math 427, Autumn 2019

Theorem: assume f is continuous

Suppose that f maps $\mathbb{C}$ to either $\mathbb{C}$ or $\mathbb{R}$. If E is open, then the pre-image $f^{-1}(E) = \{z : f(z) \in E\}$ is an open subset of $\mathbb{C}$.

Proof. Show if $w \in f^{-1}(E)$, then $D_\delta(w) \subset f^{-1}(E)$ some $\delta > 0$.

- Since E open, $E \supset D_\epsilon\big(f(w)\big)$, $\epsilon > 0$.
- f continuous, so $f(z) \in D_\epsilon\big(f(w)\big) \subset E$ if $|w - z| < \delta$.

Example: $\{z : |z^3 + z| < 1\}$ is open.

Fact

Suppose $E \subset \mathbb{C}$ is open, and f is a function from E to $\mathbb{C}$. Then f is continuous if and only if the following property holds:

$f^{-1}(U)$ is an open subset of E whenever U is open.

A multi-valued function f on $E \subset \mathbb{C}$ assigns a **set** of complex values to each $z \in E$, i.e. $f(z)$ is a set of complex numbers.

Examples:

- $\log z = \log|z| + i\arg(z)$ with domain $E = \mathbb{C}\backslash\{0\}$.
  The multiple values of $\log z$ differ by $k2\pi i$
- $\sqrt{z}$ assigns to $z \in \mathbb{C}$ the numbers w with $w^2 = z$.
  If $z \neq 0$, $\sqrt{z}$ has exactly two values, of the form $\{w, -w\}$.
- $\sqrt{z^2 - 1}$ assigns to $z \in \mathbb{C}$ the $w \in \mathbb{C}$ with $w^2 = z^2 - 1$.

A **branch** of a multi-valued function f on $E \subset \mathbb{C}$ is a function that assigns to each $z \in E$ one value from $f(z)$.

Principal branch of $z^{\frac{1}{n}}$.

The **principal branch** of $\log z$ is $\log|z| + i\arg_{(-\pi,\pi]}(z)$, $z \neq 0$.

- For any branch of $\log z$,

$$\left(e^{\frac{1}{n}\log z}\right)^n = e^{\log z} = z$$

The **principal branch** of $z^{\frac{1}{n}} = \sqrt[n]{z}$, for $z \neq 0$, is the function

$$e^{(\log|z|+i\arg_{(-\pi,\pi]}(z))/n} = |z|^{\frac{1}{n}}\, e^{\frac{i}{n}\arg_{(-\pi,\pi]}(z)}$$

- Gives unique solution to $w^n = z$ such that $\arg(w) \in (-\frac{\pi}{n}, \frac{\pi}{n}]$
- The principal branch of $z^{\frac{1}{n}}$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$.

Two branches for the square root of $z^2 - 1$.

- Consider $\sqrt{z^2 - 1}$; $\sqrt{\cdot}$ the principal branch of square root.
  Let $E = \{z : z^2 - 1 \in \mathbb{C} \setminus (-\infty, 0]\}$. E is an open set, and

$$E = \mathbb{C} \setminus \{ [-1, 1] \cup i\mathbb{R} \}$$

  Each point $z \in (-1, 1) \cup i\mathbb{R}$ is a point of discontinuity.

- $w = \sqrt{z - 1}\,\sqrt{z + 1}$ also solves $w^2 = z^2 - 1$.
  By composition, this is continuous on $F = \mathbb{C} \setminus (-\infty, 1]$.
- In fact, it is continuous on $\mathbb{C} \setminus [-1, 1]$, and
  each point in $(-1, 1)$ is a point of discontinuity.