Lecture 5: The complex logarithm function

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Recall: polar form for $z \neq 0$: $\quad z = r\,e^{i\theta}\,, \quad r = |z|$.

θ is called *the argument* of z, we write $\theta = \arg(z)$.

Notation. If I is a half-open interval of length 2π, and $z \neq 0$,

$\arg_{\mathrm{I}}(z)$ is the choice of $\arg(z)$ with $\arg_{\mathrm{I}}(z) \in \mathrm{I}$

Main examples: $\mathrm{I} = [0, 2\pi)$ or $\mathrm{I} = (-\pi, \pi]$.

$$\arg_{[0,2\pi)}(-i) = \tfrac{3}{2}\pi\,, \qquad \arg_{(-\pi,\pi]}(-i) = -\tfrac{1}{2}\pi$$

- $\arg_{\mathrm{I}}(z)$ has a *cut line* where the value jumps by 2π.

Complex logarithms

To solve $e^w = z$, for $z \neq 0$:

- Let $w = u + iv$, so $e^w = e^u e^{iv}$, and write $z = |z|\, e^{i\theta}$.

$$e^u e^{iv} = |z|\, e^{i\theta} \qquad \Leftrightarrow \qquad e^u = |z|\,, \quad e^{iv} = e^{i\theta}\,.$$

- Infinitely many solutions w: $u = \log|z|\,,\ \ v = \theta + 2\pi k$

Describe all solutions by: $\log(z) \;=\; \log|z| + i\arg(z)$

Examples:

$$\log(2i) \;=\; \log(2) + i\tfrac{\pi}{2} + i2\pi k$$

$$\log(-3) \;=\; \log(3) + i\pi + i2\pi k$$

$$\log(2+5i) \;=\; \log\left(\sqrt{29}\right) + i\arctan(\tfrac{5}{2}) + i2\pi k$$

Branches of the logarithm function

A **branch** of $\log z$ is defined by fixing the range I of $\arg(z)$:

$$\log z \;=\; \log|z| + i\arg_{\mathrm{I}}(z) \quad \text{satisfies} \quad \mathrm{Im}\big(\log(z)\big) \in \mathrm{I}$$

A branch of $\log z$ jumps by $2\pi i$ as z crosses the cut line.

For any branch of $\log z$, and $z, w \neq 0$:

- $e^{\log z} = z$
- $\log(e^z) = z + i2\pi k$ for some k.
- $\log(zw) \;=\; \log z + \log w + i2\pi k$ for some k.

Branches of the n-th root function

Can similarly define a **branch** of $z^{1/n}$ by fixing range of $\arg(z)$.

Example 1: Let $\arg_{(-\pi,\pi]}(z)$ be the value of $\arg(z)$ in $(-\pi,\pi]$,

$$\text{define}: \qquad z^{1/2} = |z|^{1/2}\, e^{i \arg_{(-\pi,\pi]}(z)/2}$$

- This choice of square root has positive real part.
- It is discontinuous across negative real axis.

Example 2: Let $\arg_{[0,2\pi)}(z)$ be the value of $\arg(z)$ in $[0,2\pi)$,

$$\text{define}: \qquad z^{1/2} = |z|^{1/2}\, e^{i \arg_{[0,2\pi)}(z)/2}$$

- This choice of square root has positive imaginary part.
- It is discontinuous across positive real axis.