Lecture 27: Essential singularities; Harmonic functions

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Theorem: assume f analytic on $D_r(z_0) \setminus \{z_0\}$

If f has an essential singularity at z_0, then for all $w \in \mathbb{C}$ and all $\delta > 0$, there is a $z \in D_r(z_0) \setminus \{z_0\}$ so that $|f(z) - w| < \delta$.

Proof by contradiction. If not, there is a $w \in \mathbb{C}$ and $c > 0$:

$$|f(z) - w| > c \quad \Rightarrow \quad \left| \frac{1}{f(z) - w} \right| < c^{-1} \quad \text{for} \;\; z \in D_r(z_0) \setminus \{z_0\}.$$

- This implies $\big(f(z) - w\big)^{-1} = g(z)$ has removable singularity.
- $f(z) = g(z)^{-1} + w$ has a pole if $g(z_0) = 0$,

  or a removable singularity if $g(z_0) \neq 0$.

Example: $e^{\frac{1}{z}}$ has an essential singularity at 0.

Claim: if $w \neq 0,\ r > 0$, there is a z with $|z| < r$ and $e^{\frac{1}{z}} = w$.

Proof. Given any $w \neq 0$ and $r > 0,$ can choose k so that

$$\left|\log w + 2\pi i k\right|^{-1} < r$$

Choose $z = \left(\log w + 2\pi i k\right)^{-1}$. □

Observation: For $z \neq 0,$ there is a convergent expansion

$$e^{\frac{1}{z}} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{1}{z}\right)^{k} = \sum_{k=-\infty}^{0} a_k\, z^k\,, \qquad a_k = \frac{1}{(-k)!}$$

Laurent expansion about an isolated singularity

Fact: assume f is analytic on $D_r(z_0) \setminus \{z_0\}$

There exists an expansion, convergent when $0 < |z - z_0| < r$,

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k = \sum_{k=0}^{\infty} a_k (z - z_0)^k + \sum_{k=1}^{\infty} \frac{a_{-k}}{(z - z_0)^k}$$

Removable: $a_k = 0$ if $k < 0$, $\qquad f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$

Pole order m: $a_k = 0$ if $k < -m$, $\qquad f(z) = \sum_{k=-m}^{\infty} a_k (z - z_0)^k$

Essential singularity: infinitely many $a_k \neq 0$ with $k < 0$.

Harmonic functions

Definition

A function $u(x,y)$ on an open set $E \subset \mathbb{R}^2$ is harmonic if:

$$\partial_x^2 u(x,y) + \partial_y^2 u(x,y) = 0 \quad \text{for all} \quad (x,y) \in E.$$

Key fact: If $f = u + iv$ is analytic, then u and v are harmonic,

The real and imaginary parts of an analytic function are harmonic.

The proof is an easy consequence of the

Cauchy-Riemann equations

$$\partial_x u(x,y) = \partial_y v(x,y)\,, \qquad \partial_y u(x,y) = -\partial_x v(x,y)\,.$$

Theorem: assume $E \subset \mathbb{C}$ is open, convex

If u is a real-valued, harmonic function on E, then there is a real-valued, harmonic function v on E so that $u + iv$ is analytic.

Proof. The function $g = \partial_x u - i\partial_y u$ is analytic on E, by the Cauchy-Riemann equations. Its anti-derivative $f(z)$ is analytic:

$$f(z) \;=\; u(x_0, y_0) \;+\; \int_{z_0}^{z} g(z)\, dz\,, \qquad z_0 = x_0 + iy_0 \in E.$$

The real part of $f(z)$ equals $u(x, y)$, since

$$\begin{aligned} f(z) \;=\; u(x_0, y_0) \;+\; \int_{z_0}^{z} \Big(\partial_x u\, dx \;+\; \partial_y u\, dy\Big) \\ + i \int_{z_0}^{z} \Big(\partial_x u\, dy \;-\; \partial_y u\, dx\Big) \end{aligned}$$

Choose $\quad v = \displaystyle\int_{z_0}^{z} \Big(\partial_x u\, dy \;-\; \partial_y u\, dx\Big)$