Lecture 24: Zeroes of analytic functions

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Math 427, Autumn 2019
Assume $f(z)$ analytic on $E \subset \mathbb{C}$, and $f(z_0) = 0$. If $|z - z_0| < R$:

$$f(z) = \sum_{k=1}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{f^{(k)}(z_0)}{k!}$$

Two possibilities:

- If $a_k = 0$ for every $k$, then $f(z) = 0$ for $|z - z_0| < R$.

- If for some $m$, we have $a_m \neq 0$ but $a_k = 0$ when $k < m$, we say that $f(z)$ has a zero of order $m$ at $z_0$. Equivalently:

  - $f(z)$ has a zero of order $m$ at $z_0$ if:

    $$f^{(m)}(z_0) \neq 0, \quad \text{and} \quad f^{(k)}(z_0) = 0 \text{ for } k < m.$$
Examples

- \( \sin(z) \) has a zero of order 1 at \( z_0 = 0 \):
  \[
  \sin(z) = z - \frac{1}{3!} z^3 + \cdots, \quad \text{so} \quad a_0 = 0, \quad a_1 \neq 0.
  \]
  Can also check: \( \sin(z) = 0, \quad \sin'(0) = \cos(0) = 1 \neq 0 \).

- \( \sin(z) \) has a zero of order 1 at \( z_0 = k\pi \).

- \( z^3 - 1 \) has a zero of order 1 at \( z_0 = 1 \):
  \[
  z^3 - 1 = 0 \quad \text{when} \quad z = 1, \quad (z^3 - 1)' = 3z^2 = 3 \quad \text{when} \quad z = 1.
  \]

- \( e^z - z - 1 \) has a zero of order 2 at \( z_0 = 0 \):
  \[
  e^z - z - 1 = \frac{1}{2!} z^2 + \frac{1}{3!} z^3 + \cdots
  \]
Theorem: assume $f$ analytic on $E \subset \mathbb{C}$

If $f(z)$ has a zero of order $m$ at $z_0$, there is $g(z)$ analytic on $E$:

$$f(z) = (z - z_0)^m g(z), \quad \text{where} \quad g(z_0) \neq 0.$$

**Proof.** For $|z - z_0| < R$ we can write:

$$f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k = (z - z_0)^m \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k.$$

Define:

$$g(z) = \begin{cases} 
\sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k, & |z - z_0| < R, \\
\frac{f(z)}{(z - z_0)^m}, & z \neq z_0.
\end{cases}$$

If $f(z)$ has a zero of order $m$ at $z_0$, then $\frac{f(z)}{(z - z_0)^m}$, defined on the set $E \setminus \{z_0\}$, extends to an analytic function on $E$. 
Theorem: L’Hôpital’s rule

If \( f(z_0) = g(z_0) = 0 \), then
\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}
\]

**Proof.** Unless the order of zeroes of \( f \) and \( g \) at \( z_0 \) are the same, then both limits are either 0 or \( \infty \). If \( f \) and \( g \) have zero order \( m \):

\[
f(z) = \sum_{k=m}^{\infty} a_k (z - z_0)^k, \quad g(z) = \sum_{k=m}^{\infty} b_k (z - z_0)^k
\]

\[
f'(z) = \sum_{k=m}^{\infty} k a_k (z - z_0)^{k-1}, \quad g'(z) = \sum_{k=m}^{\infty} k b_k (z - z_0)^{k-1}
\]

\[
\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{a_m}{b_m}, \quad \lim_{z \to z_0} \frac{f'(z)}{g'(z)} = \frac{m a_m}{m b_m} = \frac{a_m}{b_m}
\]
Zeroes of analytic functions are isolated

**Theorem:** Suppose $f$ is analytic on connected open set $E \subset \mathbb{C}$.

If $f(z_0) = 0$, and $f$ is not identically 0, then for some $r > 0$:

$$f(z) \neq 0 \text{ if } 0 < |z - z_0| < r.$$

**Proof.** Write $f(z) = (z - z_0)^m h(z)$, $h(z_0) \neq 0$. By continuity:

$$h(z) \neq 0 \text{ if } |z - z_0| < r, \text{ for some } r > 0.$$

**Application.** If $\{z_k\} \subset E$ is a sequence with $\lim_{k \to \infty} z_k = z_0 \in E$, and $f(z_k) = g(z_k)$ for all $k$, then $f(z) = g(z)$ on $E$.

**Proof.** Let $h(z) = f(z) - g(z)$. Then $h(z_0) = 0$ by continuity. For every $r > 0$, is some $z_k$ with $|z_k - z_0| < r$, and $h(z_k) = 0$, so $h(z)$ must be identically 0.