The function \( f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \) is analytic on \( D_R(z_0) \),
and \( f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1} \).

Consequence: if \( f'(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k \) on \( D_R(z_0) \), then

\[
f(z) = f(z_0) + \sum_{k=1}^{\infty} \frac{b_{k-1}}{k} (z - z_0)^k, \quad z \in D_R(z_0)
\]

- radius convergence of \( f(z) = \) radius convergence of \( f'(z) \)
Example: \( f(z) = \log z, \quad f'(z) = \frac{1}{z} \)

Expansion of \( z^{-1} \) on \(|z - 1| < 1|:

\[
\frac{1}{z} = \frac{1}{1 + (z - 1)} = \sum_{k=0}^{\infty} (-1)^k (z - 1)^k
\]

Expansion of \( \log z \) (principal branch) on \(|z - 1| < 1|:

\[
\log z = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (z - 1)^k
\]
Example: \( f(z) = \arctan z \), \( f'(z) = \frac{1}{1 + z^2} \)

Expansion of \( \frac{1}{1 + z^2} \) on \(|z| < 1\):

\[
\frac{1}{1 + z^2} = \sum_{k=0}^{\infty} (-1)^k z^{2k}
\]

Expansion of \( \arctan z \) (principal branch) on \(|z| < 1\):

\[
\arctan z = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} z^{2k+1}
\]
Power series expansions of analytic functions

If $f(z)$ is analytic on an open set containing $D_r(z_0)$, then $f(z)$ has a convergent power series expansion for $z \in D_r(z_0)$,

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad a_k = \frac{f^{(k)}(z_0)}{k!}$$

Remark: $D_r(z_0)$ is an open set containing $D_r(z_0)$.

Remark: The power series expansion of $f$ may converge on a larger set than the largest $D_r(z_0)$ contained in $E$. 
**Fact:** The power series expansion of \( \log z \) about \( z_0 \) has radius of convergence \( R = |z_0| \), for \( z_0 \neq 0 \), and any branch of \( \log z \).

**Proof.** The radius of convergence for \( \log z \) about \( z_0 \) is the same as the radius of convergence for its derivative, \( (\log z)' = z^{-1} \).

The function \( z^{-1} \) is analytic on \( \mathbb{C} \setminus \{0\} \supset D_{|z_0|}(z_0) \).

- For any branch of \( \log z \), its power series expansion at \( z_0 \) is
  \[
  \log(z_0) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \frac{(z - z_0)^k}{z_0^k}
  \]

- If \( D_{|z_0|}(z_0) \) extends across the cut line, the series expansion does not agree with that branch across the cut line.
Theorem: Cauchy’s estimates

If \( f(z) \) is analytic on an open set containing \( \overline{D}_r(z_0) \), then

\[
\frac{|f^{(k)}(z_0)|}{k!} \leq \frac{M}{r^k}, \quad M = \max_{|w-z_0|=r} |f(w)|
\]

Proof. The coefficients \( a_k = f^{(k)}(z_0)/k! \) are given by

\[
a_k = \frac{1}{2\pi i} \int_{|w-z_0|=r} \frac{f(w)}{(w-z_0)^{k+1}} \, dz
\]

Use Theorem 2.4.9: \( \ell(\gamma) = 2\pi r \), \( |f(w)/(w-z_0)^{k+1}| \leq M/r^{k+1} \)

Remark. This proves that \( \sum_{k=0}^{\infty} a_k(z-z_0)^k \) converges on \( D_r(z_0) \); we already knew this from the proof that it equals \( f(z) \) there.