Lecture 12: Contour integrals II

Hart Smith

Department of Mathematics
University of Washington, Seattle

Math 427, Autumn 2019

Contour integral

Definition

If $f(z)$ is a continuous function on $E \subset \mathbb{C}$, and $\gamma(t) : [a, b] \to E$ is a smooth curve, the contour integral of f over γ is

$$\int_\gamma f(z)\, dz = \int_a^b f(\gamma(t))\, \gamma'(t)\, dt$$

Intuition: think of plugging in $z = \gamma(t), \ \ dz = \dfrac{d\gamma(t)}{dt}\, dt$.

By Fundamental Theorem of Calculus: if f is analytic on E

$$\int_\gamma f'(z)\, dz = f(\gamma(b)) - f(\gamma(a))$$

Relation to other line integrals

For $\gamma = \big(x(t), y(t)\big)$ a path in $\mathbb{R}^2$, $u(x, y)$ a continuous function

$$\int_\gamma u(x,y)\,dx \;=\; \int_a^b u\big(x(t),y(t)\big)\,x'(t)\,dt$$

$$\int_\gamma u(x,y)\,dy \;=\; \int_a^b u\big(x(t),y(t)\big)\,y'(t)\,dt$$

If $f(x+iy) \;=\; u(x,y) + i\,v(x,y)$, then

$$\int_\gamma f(z)\,dz \;=\; \int_\gamma \Big(u(x,y) + i\,v(x,y)\Big)\,(dx + i\,dy)$$

$$= \int_\gamma \Big(u\,dx \;-\; v\,dy\Big) \;+\; i\int_\gamma \Big(v\,dx + u\,dy\Big)$$

Example

- Let $\gamma(t) = e^{it}$ for $t \in [0, \frac{\pi}{2}]$. Calculate $\displaystyle\int_\gamma z\,dz$.

Using the definition we get

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} e^{it}\, ie^{it}\, dt &= i \int_0^{\frac{\pi}{2}} e^{2it}\, dt \\
&= i \int_0^{\frac{\pi}{2}} \cos(2t) + i \sin(2t)\, dt \\
&= i \left(\tfrac{1}{2} \sin(2t) - i\, \tfrac{1}{2} \cos(2t) \right) \Big|_0^{\frac{\pi}{2}} \\
&= -1
\end{aligned}$$

Get same result if write $z = \left(\frac{1}{2} z^2\right)'$, which by F.T.C. gives

$$\int_\gamma \left(\tfrac{1}{2} z^2\right)' dz = \tfrac{1}{2}(e^{i\pi/2})^2 - \tfrac{1}{2}(e^{i0})^2 = -\tfrac{1}{2} - \tfrac{1}{2}$$

Example

- Let $\gamma(t) = e^{it}$ for $t \in [0, \frac{\pi}{2}]$. Calculate $\displaystyle\int_\gamma \overline{z}\, dz$.

Using the definition we get

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \overline{e^{it}}\, ie^{it}\, dt &= i \int_0^{\frac{\pi}{2}} e^{-it}\, e^{it} \\ &= i \int_0^{\frac{\pi}{2}} 1\, dt \\ &= i\, \tfrac{\pi}{2} \end{aligned}$$

Can't write $\overline{z} = f'(z)$ for analytic function $f(z)$, can't use F.T.C.

A piecewise example

Let $\gamma(t) = \begin{cases} t\,, & 0 \le t \le 1\,, \\ 1 + i(t-1)\,, & 1 \le t \le 2\,, \end{cases}$ goes from 0 to $1 + i$.

Need to separate an integral over γ into smooth pieces:

$$\begin{aligned} \int_\gamma e^z\,dz &= \int_0^1 e^t\,dt + \int_1^2 e^{1+i(t-1)}\,i\,dt \\ &= e^t\,\Big|_0^1 + e^{1-i}\int_1^2 e^{it}\,i\,dt \\ &= e^1 - e^0 + e^{1-i}\left(e^{2i} - e^i\right) \\ &= e^{1+i} - e^0 \end{aligned}$$

Independence of parameter

Theorem

Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a smooth path, and that $\alpha(t) : [c, d] \to [a, b]$ is a smooth function, with $\begin{cases} \alpha(c) = a\,, \\ \alpha(d) = b\,. \end{cases}$

If $\mu : [c, d] \to \mathbb{C}$ is the path $\mu(t) = \gamma(\alpha(t))$, then

$$\int_\mu f(z)\, dz = \int_\gamma f(z)\, dz\,.$$

Proof. By the change of variables formula for integrals

$$\int_c^d f(\gamma(\alpha(t)))\, \gamma'(\alpha(t))\, \alpha'(t)\, dt = \int_a^b f(\gamma(s))\, \gamma'(s)\, ds$$

- Important that $\mu(c) = \gamma(a)$, $\mu(d) = \gamma(b)$: same direction!
- For piecewise smooth γ: if $\gamma = \gamma_1 \cup \cdots \cup \gamma_n$, can take a new parameterization μ_j for each γ_j.