Lecture 11: Contour integrals

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Math 427, Autumn 2019

Curves in the complex plane

Definition

A curve is a continuous map $\gamma(t) : [a, b] \to \mathbb{C}$, some $[a, b] \subset \mathbb{R}$.

Example: A straight line curve from z to w:

$$\gamma(t) = (1 - t)z + tw\,, \qquad t \in [0, 1]$$

A curve that moves counterclockwise around the unit circle in $\mathbb{C}$:

$$\gamma(t) = e^{it}\,, \qquad t \in [0, 2\pi]$$

A curve that moves counterclockwise around the unit square:

$$\gamma(t) = \begin{cases} t\,, & 0 \le t \le 1 \\ 1 + i(t-1) & 1 \le t \le 2 \\ (3-t) + i & 2 \le t \le 3 \\ (4-t)i & 3 \le t \le 4 \end{cases}$$

Piecewise smooth curves

- A curve is *smooth* if $\gamma'(t)$ is continuous.

Remark: $\gamma'(t)$ is the usual derivative of $\gamma(t)$ as a function

$$\gamma'(t) = \frac{d\gamma(t)}{dt} = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

with one-sided derivatives at endpoints $t = a$ and $t = b$.

- A curve is *piecewise smooth* if one can partition

$$a = a_0 < a_1 < \cdots < a_n = b$$

such that γ is smooth on each subinterval $[a_k, a_{k+1}]$.

Chain rule for complex curves

If $f(z)$ is analytic on E, where E contains the image of γ, then

$$\frac{df(\gamma(t))}{dt} = f'(\gamma(t))\,\gamma'(t)$$

Proof. Same as proof of standard chain rule: take limit of

$$\frac{f(\gamma(t+h)) - f(\gamma(t))}{h} = \frac{f(\gamma(t+h)) - f(\gamma(t))}{\gamma(t+h) - \gamma(t)}\,\frac{\gamma(t+h) - \gamma(t)}{h}$$

Corollary: Fundamental Theorem of Calculus

If $\gamma(t)$ is a piecewise smooth curve, and f is analytic on a set E containing the image of γ, and $f'(z)$ is continuous on E, then

$$\int_a^b f'(\gamma(t))\,\gamma'(t)\,dt = f(\gamma(b)) - f(\gamma(a))$$

Integrals of complex valued functions over $[a, b]$

If $f(t) = u(t) + iv(t)$ continuous on $[a, b]$, can identify

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Agrees with writing integral as limit of Riemann sums.

Similar properties hold as for real integrals:

$$\int_a^b \alpha\, f(t)\,dt = \alpha\int_a^b f(t)\,dt\,, \quad \alpha \in \mathbb{C}$$

$$\left|\int_a^b f(t)\,dt\right| \leq \int_a^b |f(t)|\,dt$$

Contour integral

Definition

If $f(z)$ is a continuous function on $E \subset \mathbb{C}$, and $\gamma(t) : [a, b] \to E$ is a smooth (or piecewise smooth) curve, we define

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt$$

Intuition: think of plugging in $z = \gamma(t)$, $dz = \frac{d\gamma(t)}{dt}\,dt$.

By above corollary:

$$\int_\gamma f'(z)\,dz = f(\gamma(b)) - f(\gamma(a))$$