Cauchy-Riemann equations.

We will write $w = x + iy$, and express

$$f(x + iy) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real-valued functions on $\mathbb{R}^2$.

Consider $z = w + h$, where $h$ is a real number. Then

$$\frac{f(z) - f(w)}{z - w} = \frac{u(x + h, y) - u(x, y)}{h} + i \frac{v(x + h, y) - v(x, y)}{h}$$

If $f$ is differentiable at $w$, taking the limit as $h \to 0$ gives

$$f'(x + iy) = \partial_x u(x, y) + i \partial_x v(x, y).$$
Consider $z = w + ih$, where $h$ is a real number. Then

$$
\frac{f(z) - f(w)}{z - w} = \frac{u(x, y + h) - u(x, y)}{ih} + i \frac{v(x, y + h) - v(x, y)}{ih}
$$

If $f'(x + iy)$ exists, then taking the limit as $h \to 0$ gives

$$
f'(x + iy) = -i \partial_y u(x, y) + \partial_y v(x, y).
$$

Thus: $\partial_x u(x, y) + i \partial_x v(x, y) = -i \partial_y u(x, y) + \partial_y v(x, y)$, so

Cauchy-Riemann equations: if $f = u + iv$ is analytic, then

$$
\partial_x u(x, y) = \partial_y v(x, y), \quad \partial_y u(x, y) = -\partial_x v(x, y).
$$
Theorem: suppose $f(x + iy) = u(x, y) + iv(x, y)$

If $u$ and $v$ are differentiable on $E$, then $f$ is analytic on $E$ if

$$\partial_x u(x, y) = \partial_y v(x, y), \quad \partial_y u(x, y) = -\partial_x v(x, y).$$

Example:

$$e^{x+iy} = e^x (\cos y + i \sin y)$$

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y.$$
Lemma

For any branch of $\arg(x, y) = \arg(x + iy)$, on $\mathbb{C} \setminus \{\text{cut-line}\}$

$$\frac{\partial_x \arg(x, y)}{x^2 + y^2} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial_y \arg(x, y)}{x^2 + y^2} = \frac{x}{x^2 + y^2}.$$

Proof. It suffices to prove it near each point for some branch, since different branches differ by a constant $2\pi k$.

- For $x > 0$ choose $\arg(x, y) = \arctan(y/x)$.
- For $x < 0$ choose $\arg(x, y) = \arctan(y/x) + \pi$.
- For $y > 0$ choose $\arg(x, y) = \arccot(x/y)$.
- For $y < 0$ choose $\arg(x, y) = \arccot(x/y) + \pi$. 
Derivative of \( \log z \)

**Principal branch:** \( \log(z) = \log|z| + i \arg_{(-\pi,\pi]}(z) \)

\[
\begin{align*}
  u(x, y) &= \frac{1}{2} \log(x^2 + y^2), \\
  v(x, y) &= \arg_{(-\pi,\pi]}(x, y).
\end{align*}
\]

C-R holds, partial derivatives are continuous on \( \mathbb{C} \setminus (-\infty, 0] \), so

\[
(\log z)' = \partial_x u + i \partial_x v = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} = \frac{1}{x + iy}
\]

\[
(\log z)' = \frac{1}{z}
\]

This rule holds for every branch of \( \log z \), off its cut-line.
Principal branch: $z^{\frac{1}{2}} = e^{\frac{1}{2} \log z}$ (principal branch of log)

By chain rule:

$$
(z^{\frac{1}{2}})' = e^{\frac{1}{2} \log z} \cdot \frac{1}{2} (\log z)' = \frac{1}{2} z^{\frac{1}{2}} z^{-1}
$$

Writing $z^{-1} = e^{-\log z}$, we see $z^{\frac{1}{2}} z^{-1} = e^{-\frac{1}{2} \log z}$,

$$
(z^{\frac{1}{2}})' = \frac{1}{2} z^{-\frac{1}{2}}
$$

where $z^{-\frac{1}{2}}$ is also the principal branch.