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Text: *Complex Variables*, Joseph Taylor (AMS, 2011)

Office Hours: Padelford C-447, MWF 4:00 – 5:00 pm.

Grading:

- **Midterm, Friday November 1**: 30%
- **Final Exam, Thursday December 12**: 50%
- **Weekly Homework**: 20%

Lecture 1: Complex Arithmetic

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Complex Numbers

Quick definition of a complex number:

Something of the form $x + iy$, where x and y are real numbers.

- x is called the *real* part: $x = \text{Re}(x + iy)$

- y is the *imaginary* part: $y = \text{Im}(x + iy)$

Rules for complex arithmetic:

Apply usual rules for addition and multiplication, plus the rule

$$i^2 = -1$$

Constructive definition:

Complex number $\equiv$ pair of real numbers (x, y), and define

$$(u, v) + (x, y) = (u + x, v + y)$$

$$(u, v) \times (x, y) = (ux - vy, uy + vx)$$

Verify that rules of arithmetic hold:

- addition & multiplication are commutative, associative
- multiplication distributes over addition

Complex numbers contain the reals: identify $x \equiv (x, 0)$

Introduce notation: $i = (0, 1)$, then

- $i^2 = (0, 1) \times (0, 1) = (-1, 0) \equiv -1$
- $x + iy = (x, 0) + (0, 1) \times (y, 0) = (x, 0) + (0, y) = (x, y)$

Conjugation

Definition

If $z = x + iy$, its *conjugate* is $\overline{z} = x - iy$.

Properties:

- $\overline{z + w} = \overline{z} + \overline{w}$
- $\overline{zw} = \overline{z}\,\overline{w}$
- $z + \overline{z} = 2\,\mathrm{Re}(z)$, $z - \overline{z} = 2i\,\mathrm{Im}(z)$
- If $z = x + iy$, then $z\,\overline{z} = x^2 + y^2$
- If $z = x + iy$, $w = u + iv$ then

$$\mathrm{Re}(z\overline{w}) = xu + yv = (x, y) \cdot (u, v) \quad \leftarrow \text{dot product}$$

Modulus and the triangle inequality

Definition

If $z = x + iy$, its *modulus* is $|z| = \sqrt{x^2 + y^2} = \sqrt{z\,\bar{z}}$.

- If $z = x$ is real, then $|z| = |x|$ is the absolute value of x.
- $|\bar{z}| = |z|$, $\quad |\operatorname{Re}(z)| \le |z|$, $|\operatorname{Im}(z)| \le |z|$.
- $|zw| = |z||w|$.

$$|zw|^2 = zw\overline{zw} = z\,\bar{z}\,w\bar{w} = |z|^2|w|^2$$

- $|z + w| \le |z| + |w|$.

$$\begin{aligned} |z+w|^2 = (z+w)(\bar{z}+\bar{w}) &= |z|^2 + z\bar{w} + \bar{z}w + |w|^2 \\ &= |z|^2 + 2\operatorname{Re}(z\bar{w}) + |w|^2 \\ &\le |z|^2 + 2|z||w| + |w|^2 \\ &= (|z| + |w|)^2 \end{aligned}$$

Complex division

$$z \neq 0 : \qquad z^{-1} = \frac{\bar{z}}{|z|^2} = \frac{x}{x^2+y^2} + i\,\frac{-y}{x^2+y^2}$$

Complex numbers are a Field:

- Can add, subtract, multiply, divide (except by 0)
- Addition and Multiplication are associative, commutative
- Multiplication distributes over Addition

Unlike the reals, cannot order the complex numbers

- No suitable notion of $z > w$