Name: Solutions
Mathematics
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MIDTERM 2 REVIEW SOLUTIONS

Problem 1. Determine a suitable form for $Y_p(t)$, the particular solution, if the method of undetermined coefficients is to be used. *You don't need to find the coefficients, just write down the form.*

(a) $y'' + 3y' = 2t^3 + t^2e^{-3t} + 4\sin(3t)$

Solutions:

(1) Find the solution to the homogeneous equation $y'' + 3y' = 0$:
The characteristic polynomial is given by

$$r^2 + 3r = 0 \quad \Rightarrow \quad r(r+3) = 0.$$

Hence, the roots of the polynomial are given by $r = 0$ and $r = -3$. These are distinct real roots so the solution is

$$y = c_1e^{0t} + c_2e^{-3t} = c_1 + c_2e^{-3t}.$$

(2) Break the problem into separate differential equations:

$$\begin{aligned} &(a) \qquad y'' + 3y' = 2t^3 \\ &(b) \qquad y'' + 3y' = t^2e^{-3t} \\ &(c) \qquad y'' + 3y' = 4\sin(3t) \end{aligned}$$

(3) Guess the form for each of the differential equations:

$$\begin{aligned} \text{For } (a) \qquad & Y_p = t^s\big(A_3t^3 + A_2t^2 + A_1t + A_0\big) \\ \text{For } (b) \qquad & Y_p = t^s\big(B_2t^2 + B_1t + B_0\big)e^{-3t} \\ \text{For } (c) \qquad & Y_p = t^s\Bigg(C\sin(3t) + D\cos(3t)\Bigg) \end{aligned}$$

(4) Determine s:

For (a):

- If $s = 0$, then $Y_p = (A_3t^3 + A_2t^2 + A_1t + A_0)$ which contains a multiple of the homogeneous equation
- If $s = 1$, then $Y_p = t(A_3t^3 + A_2t^2 + A_1t + A_0)$ which doesn't contain a multiple

Therefore, $Y_p = t(A_3t^3 + A_2t^2 + A_1t + A_0)$

For (b):

- If $s = 0$, then $Y_p = (B_2t^2 + B_1t + B_0)e^{-3t}$ which contains a multiple of the homogeneous equation
- If $s = 1$, then $Y_p = t(B_2t^2 + B_1t + B_0)e^{-3t}$ which doesn't contain a multiple

Therefore, $Y_p = t(B_2t^2 + B_1t + B_0)e^{-3t}$

For (c):

- If $s = 0$, then $Y_p = C\sin(3t) + D\cos(3t)$ which is not a multiple.

Therefore, $Y_p = C\sin(3t) + D\cos(3t)$.

The form of the particular solution is then

$$\begin{aligned} \text{For } (a) \quad & Y_p = t^1\big(A_3t^3 + A_2t^2 + A_1t + A_0\big) \\ \text{For } (b) \quad & Y_p = t^1\big(B_2t^2 + B_1t + B_0\big)e^{-3t} \\ \text{For } (c) \quad & Y_p = C\sin(3t) + D\cos(3t) \end{aligned}$$

(b) $y'' + 2y' + 2y = 3e^{-t} + e^{-t}\cos(2t) + 4t^2e^{-t}\sin(t)$

Solutions:

(1) Find the solution to the homogeneous equation $y'' + 2y' + 2y = 0$:
The characteristic polynomial is given by

$$r^2 + 2r + 2 = 0 \quad \Rightarrow \quad \frac{-2 \pm \sqrt{4 - 4(2)}}{2} = -1 \pm i.$$

These are complex roots so the solution is

$$y = c_1e^{-t}\cos(t) + c_2e^{-t}\sin(t).$$

(2) Break the problem into separate differential equations:

$$\begin{aligned} (a) \quad & y'' + 2y' + 2y = 3e^{-t} \\ (b) \quad & y'' + 2y' + 2y = e^{-t}\cos(2t) \\ (c) \quad & y'' + 2y' + 2y = 4t^2e^{-t}\sin(t) \end{aligned}$$

(3) Guess the form for each of the differential equations:

$$\begin{aligned} \text{For } (a) \quad & Y_p = t^s Ae^{-t} \\ \text{For } (b) \quad & Y_p = t^s\big(Be^{-t}\cos(2t) + Ce^{-t}\sin(2t)\big) \\ \text{For } (c) \quad & Y_p = t^s\Big(\big(D_2t^2 + D_1t + D_0\big)e^{-t}\sin(t) + \big(F_2t^2 + F_1t + F_0\big)e^{-t}\cos(t)\Big) \end{aligned}$$

(4) Determine s:

For (a):

- If $s = 0$, then $Y_p = Ae^{-t}$ does not contains a multiple of the homogeneous equation.

Therefore, $Y_p = Ae^{-t}$

For (b):

- If $s = 0$, then $Y_p = Be^{-t}\cos(2t) + Ce^{-t}\sin(2t)$ which not contain a multiple.

Therefore, $Y_p = Be^{-t}\cos(2t) + Ce^{-t}\sin(2t)$

For (c):

- If $s = 0$, then $Y_p = \big(D_2t^2 + D_1t + D_0\big)e^{-t}\sin(t) + \big(F_2t^2 + F_1t + F_0\big)e^{-t}\cos(t)$ which is a multiple of the homogeneous so $s \neq 0$
- If $s = 1$, $Y_p = t\Big(\big(D_2t^2 + D_1t + D_0\big)e^{-t}\sin(t) + \big(F_2t^2 + F_1t + F_0\big)e^{-t}\cos(t)\Big)$ which is not a multiple of the homogeneous.

Therefore,

$$Y_p = t\Big(\big(D_2t^2 + D_1t + D_0\big)e^{-t}\sin(t) + \big(F_2t^2 + F_1t + F_0\big)e^{-t}\cos(t)\Big)$$

The form of the particular solution is then

For (a) $\quad Y_p = Ae^{-t}$

For (b) $\quad Y_p = Be^{-t}\cos(2t) + Ce^{-t}\sin(2t)$

For (c) $\quad Y_p = t\Big(\big(D_2t^2 + D_1t + D_0\big)e^{-t}\sin(t) + \big(F_2t^2 + F_1t + F_0\big)e^{-t}\cos(t)\Big)$

(c) $y'' + 2y' + y = t\cos(2t) + e^{-t}\sin(t) + e^{-t}$

Solutions:

(1) Find the solution to the homogeneous equation $y'' + 2y + y = 0$:
The characteristic polynomial is given by

$$r^2 + 2r + 1 = 0 \quad \Rightarrow \quad (r+1)^2 = 0.$$

Hence the roots are $r = -1$. This is a repeated root so the solution is

$$y = c_1 e^{-t} + c_2 t e^{-t}.$$

(2) Break the problem into separate differential equations:

$$\begin{aligned} &(a) \qquad y'' + 2y + y = t\cos(2t) \\ &(b) \qquad y'' + 2y + y = e^{-t}\sin(t) \\ &(c) \qquad y'' + 2y + y = e^{-t} \end{aligned}$$

(3) Guess the form for each of the differential equations:

$$\begin{aligned} &\text{For } (a) \qquad Y_p = t^s\big((A_1 t + A_0)\cos(2t) + (B_1 t + B_0)\sin(2t)\big) \\ &\text{For } (b) \qquad Y_p = t^s\big(Ce^{-t}\cos(t) + De^{-t}\sin(t)\big) \\ &\text{For } (c) \qquad Y_p = t^s F e^{-t} \end{aligned}$$

(4) Determine s:

For (a):

- If $s = 0$, then $Y_p = \big((A_1 t + A_0)\cos(2t) + (B_1 t + B_0)\sin(2t)\big)$ does not contains a multiple of the homogeneous equation.

Therefore, $Y_p = (A_1 t + A_0)\cos(2t) + (B_1 t + B_0)\sin(2t)$

For (b):

- If $s = 0$, then $Y_p = Ce^{-t}\cos(t) + De^{-t}\sin(t)$ which not contain a multiple.

Therefore, $Y_p = Ce^{-t}\cos(t) + De^{-t}\sin(t)$

For (c):

- If $s = 0$, then $Y_p = Fe^{-t}$ which is a multiple of the homogeneous so $s \neq 0$
- If $s = 1$, $Y_p = tFe^{-t}$ which is a multiple of the homogeneous so $s \neq 1$
- If $s = 0$, $Y_p = t^2 Fe^{-t}$ which is not a multiple of the homogeneous so $s = 2$.

Therefore, $Y_p = t^2 Fe^{-t}$

The form of the particular solution is then

$$\begin{aligned} &\text{For } (a) \qquad Y_p = Y_p = (A_1 t + A_0)\cos(2t) + (B_1 t + B_0)\sin(2t) \\ &\text{For } (b) \qquad Y_p = Ce^{-t}\cos(t) + De^{-t}\sin(t) \\ &\text{For } (c) \qquad Y_p = t^2 Fe^{-t} \end{aligned}$$

Problem 2. Find the general solution to

$$y'' - y' + y = te^t$$

Solution:

(1) Find the solution to the homogeneous equation, $y'' - y' + y = 0$. The characteristic polynomial is $r^2 - r + 1 = 0$. Hence by the quadradic formula

$$r = \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}.$$

Hence, the general solution is

$$y = c_1 e^{t/2} \cos\left(\tfrac{\sqrt{3}}{2}\right) + c_2 e^{t/2} \sin\left(\tfrac{\sqrt{3}}{2}\right)$$

(2) Guess the particular solution.

$$Y_p = t^s(A_1 t + A_0)e^t.$$

(3) Determine s:
If $s = 0$, then $Y_p = (At + A_0)e^t$ and it is not a solution to the homogeneous.

(4) Find the coefficents:
First, one computes the derivatives

$$\begin{aligned} Y_p' &= A_1 e^t + (A_1 t + A_0)e^t \\ Y_p'' &= A_1 e^t + A_1 e^t + (A_1 t + A_0)e^t. \end{aligned}$$

Plugging this into the differential equation, one has

$$te^t = y'' - y' + y = A_1 e^t + A_1 e^t + (A_1 t + A_0)e^t - \left(A_1 e^t + (A_1 t + A_0)e^t\right) + (A_1 t + A_0)e^t.$$

This leads to following system:

$$\begin{aligned} (A_1 + A_1 + A_0 - A_1 - A_0 + A_0)e^t = 0 \quad &\Rightarrow \quad A_1 + A_0 = 0 \\ (A_1 - A_1 + A_1)te^t = te^t \quad &\Rightarrow \quad A_1 = 1. \end{aligned}$$

Therefore, $A_0 = -1$.

The general solution is given by

$$y = c_1 e^{t/2} \cos\left(\tfrac{\sqrt{3}}{2}\right) + c_2 e^{t/2} \sin\left(\tfrac{\sqrt{3}}{2}\right) + (t - 1)e^t.$$

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