Name: ______________________________

Mathematics 307 L
University of Washington

November 15, 2019

MIDTERM 2 SOLUTIONS

Here are the rules:

- This exam is closed book. No note sheets, calculators, or electronic devices are allowed.
- In order to receive credit, you must **show all of your work**; to obtain full credit, you must provide mathematical justifications. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give numerical answers in exact form (for example $\ln(\frac{\pi}{3})$ or $5\sqrt{3}$ or $e^{2.5}$).
- If you need more room, use the backs of the pages and indicate that you have done so.
- This exam has 5 pages, plus a cover sheet. Please make sure that your exam is complete.

$$\cos(\alpha-\beta) = \cos\alpha\,\cos\beta + \sin\alpha\,\sin\beta$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\,\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\,\sin\frac{\alpha-\beta}{2}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\,\cos\frac{\alpha-\beta}{2}$$

Problem	Possible	Score
1	12	12
2	15	15
3	5	5
4	11	11
5	12	12
Total	55	55

Problem 1. (12 points)

(a) (9 points) Solve the following initial value problem.

$$y'' + 3y' + \tfrac{5}{2}\,y \;=\; 5\cos t, \qquad y(0) = 1, \quad y'(0) = 0.$$

(b) (3 points) Identify the steady state for your answer y, and express it in the form $y_{\text{steady}}(t) = A\,\cos(\omega t - \phi)$,

Solution. Find the roots to $r^2 + 3r + \frac{5}{2} = (r + \frac{3}{2})^2 + \frac{1}{4}$ which are $-\frac{3}{2} \pm \frac{1}{2}i$.

General solution to homogeneous equation is $y(t) = c_1 e^{-\frac{3}{2}t}\cos(\frac{1}{2}t) + c_2\, e^{-\frac{3}{2}t}\sin(\frac{1}{2}t)$.

The term $5\cos t$ is not a polynomial times a homogeneous solution, so we find a particular solution to the inhomogeneous equation by trying $Y(t) = A\cos t + B\sin t$.

Doing the math gives

$$Y'' + 3Y' + \tfrac{5}{2}Y = (\tfrac{3}{2}A + 3B)\cos t + (\tfrac{3}{2}B - 3A)\sin t$$

Setting $\frac{3}{2}A + 3B = 5$ and $\frac{3}{2}B - 3A = 0$ gives $\boxed{A = \frac{2}{3} \text{ and } B = \frac{4}{3}}$.

So the general solution to the inhomogeneous equation is

$$y(t) = c_1 e^{-\frac{3}{2}t}\cos(\tfrac{1}{2}t) + c_2\, e^{-\frac{3}{2}t}\sin(\tfrac{1}{2}t) + \tfrac{2}{3}\cos t + \tfrac{4}{3}\sin t.$$

Then $1 = y(0) = c_1 + \frac{2}{3}$ gives $\boxed{c_1 = \frac{1}{3}}$.

And $0 = y'(0) = -\frac{3}{2}c_1 + \frac{1}{2}c_2 + \frac{4}{3}$ gives $\boxed{c_2 = -\frac{5}{3}}$. So, the answer to (a) is

$$\boxed{y(t) = \tfrac{1}{3}e^{-\frac{3}{2}t}\cos(\tfrac{1}{2}t) - \tfrac{5}{3}\,e^{-\frac{3}{2}t}\sin(\tfrac{1}{2}t) + \tfrac{2}{3}\cos t + \tfrac{4}{3}\sin t}$$

The steady state solution is the same as $Y(t)$. It is of the form $A\cos(t - \phi)$ where $A = \sqrt{(\frac{2}{3})^2 + (\frac{4}{3})^2} = 2\sqrt{5}/3$, and $\phi = \tan^{-1}(2)$. (We use $\tan^{-1}$ since $\frac{2}{3} > 0$). So,

$$\boxed{y_{\text{steady}} = \tfrac{2\sqrt{5}}{3}\,\cos\big(t - \tan^{-1}(2)\big)}$$

Problem 2. (15 points) For each of the differential equations below, find the form for the particular solution, Y_p. You do NOT need to solve for the coefficients.

(a) (5 points)

$$y'' - 2y' + 10y = 2t^2\cos(3t) + \sin(3t)$$

Solution. Answer takes the form

$$\boxed{Y_p(t) = t^s\Big(\big(A_0t^2 + A_1t + A_2\big)\cos(3t) + \big(B_0t^2 + B_1t + B_2\big)\sin(3t)\Big)}$$

Roots of $r^2 - 2r + 10 = 0$ are $r = 1 \pm 3i$. The homogeneous general solution is

$$y(t) = c_1e^t\cos(3t) + c_2\,e^t\sin(3t)$$

So in this case $\boxed{s = 0}$.

(b) (5 points)

$$y'' + y' - 6y = 10\,te^{2t} + t^2e^{-t}$$

Solution. Answer takes the form $\boxed{Y_p(t) = Y_1 + Y_2}$ where

$$\boxed{Y_1(t) = t^s\big(A_0t + A_1\big)e^{2t}} \qquad \boxed{Y_2(t) = t^s\big(B_0t^2 + B_1t + B_2\big)e^{-t}}$$

Roots of $r^2+r-6 = (r-2)(r+3) = 0$ are $r = -3$ and $r = 2$. The homogeneous general solution is

$$y(t) = c_1e^{2t} + c_2\,e^{-3t}$$

So in this case $\boxed{s = 1 \text{ for } Y_1}$ and $\boxed{s = 0 \text{ for } Y_2}$.

(c) (5 points)

$$9y'' - 12y' + 4y = 4\,te^{2t/3} + t^2$$

Solution. Answer takes the form $\boxed{Y_p(t) = Y_1 + Y_2}$ where

$$\boxed{Y_1(t) = t^s\big(A_0t + A_1\big)e^{2t/3}} \qquad \boxed{Y_2(t) = t^s\big(B_0t^2 + B_1t + B_2\big)}$$

Roots of $9r^2 - 12r + 4 = (3r-2)^2 = 0$ are $r = 2/3$ repeated. The homogeneous general solution is

$$y(t) = \big(c_1t + c_2\big)e^{2t/3}$$

So in this case $\boxed{s = 2 \text{ for } Y_1}$ and $\boxed{s = 0 \text{ for } Y_2}$.

Problem 3. (5 points) A spring is observed to stretch $\frac{1}{5}$ meter when a force of 1 newton is applied to it. A viscous damper is observed to yield a resistance of 2 newtons when it is moved at a velocity of 1 meter/second.

A mass of 2 kg is hung from the spring and attached to the viscous damper. It is then pulled $\frac{1}{2}$ meter below its rest position and released with 0 initial velocity.

Write down the differential equation and initial conditions for $u(t)$, the position of the mass at time t relative to its rest position, where $u > 0$ means the mass is above the rest position. **Do not solve the equation**. **(And yes, this problem is really short.)**

Solution. $m = 2\text{kg}$, $\gamma = (2N)/(1m/s) = 2\text{kg/s}$, and $k = (1N)/(\frac{1}{5}m) = 5\text{kg/s}^2$. So, with u in meters,

$$\boxed{2u'' + 2u' + 5u = 0 \qquad u(0) = -\tfrac{1}{2}\,, \quad u'(0) = 0}$$

Problem 4. (11 points)

(a) (7 points) Solve the following equation, and sketch the graph of the solution

$$y'' + 2y' + 5y = 0\,, \qquad y(0) = 1\,, \quad y'(0) = 1\,.$$

(b) (4 points) Find the time t at which $y(t)$ has its **maximum** value for $t > 0$. You do not need to find the value of y there, but find an exact formula for t.

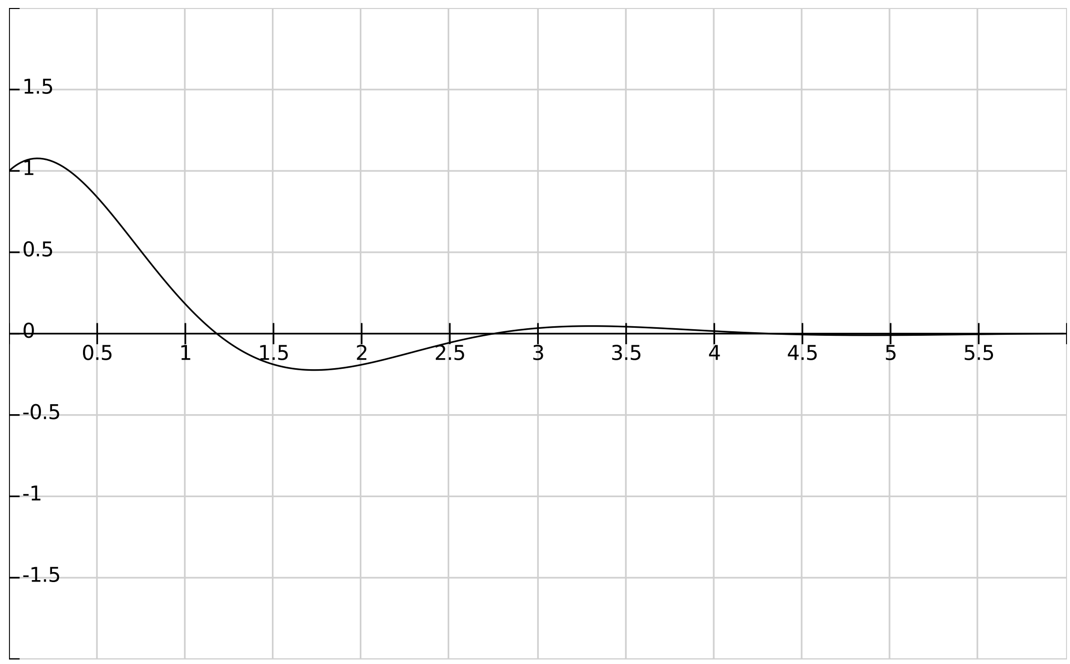
Solution. Roots of $r^2 + 2r + 5 = (r+1)^2 + 4$ are $r = -1 \pm 2i$. General solution is

$$y(t) = c_1 e^{-t}\cos(2t) + c_2\, e^{-t}\sin(2t)$$

Then $1 = y(0) = c_1$ gives $\boxed{c_1 = 1}$, and $1 = y'(0) = -c_1 + 2c_2$ gives $\boxed{c_2 = 1}$.
The answer to part (a) is

$$\boxed{y(t) = e^{-t}\Big(\cos(2t) + \sin(2t)\Big)}$$

You can write this as $\sqrt{2}\, e^{-t}\cos(2t - \frac{\pi}{4})$, though you do not need to.

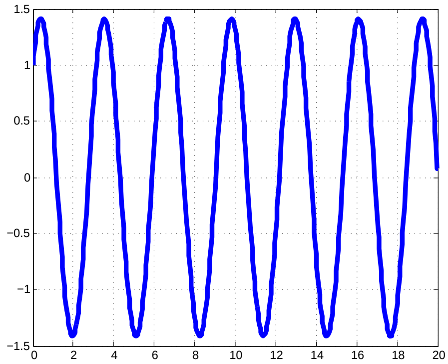


To answer (b) we set $y'(t) = 0$. We calculate

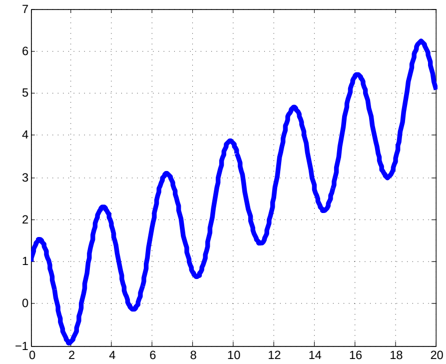
$$y'(t) = -e^{-t}\Big(\cos(2t) + \sin(2t)\Big) + e^{-t}\Big(-2\sin(2t) + 2\cos(2t)\Big)$$

Since $e^{-t} \neq 0$, setting $y'(t) = 0$ leads to $\cos(2t) = 3\sin(2t)$, or $\tan(2t) = \frac{1}{3}$.
Since the graph is initially increasing and the envelope is decreasing, the maximum is attained at the smallest positive value of t so $\tan(2t) = \frac{1}{3}$, or $\boxed{t_{\max} = \frac{1}{2}\tan^{-1}(\frac{1}{3})}$.

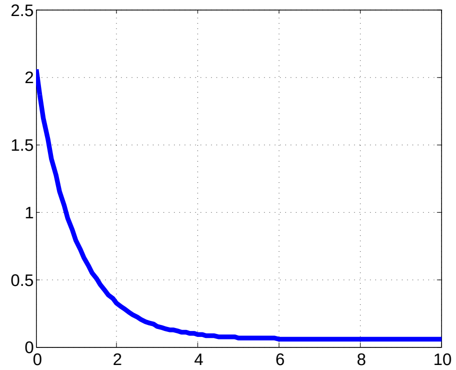
Problem 5. (12 points) Each of the 6 differential equations below has a solution that is plotted in one of the graphs. Match each of the differential equations to its solution. (Note: only 6 of the graphs will correspond to an equation.)



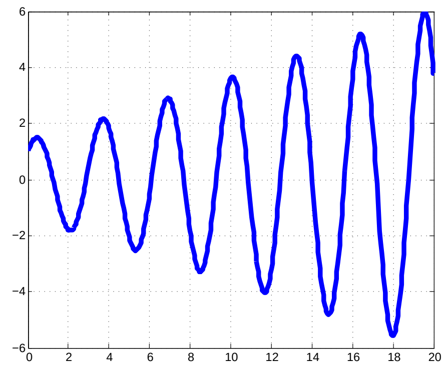
(a)



(b)



(c)



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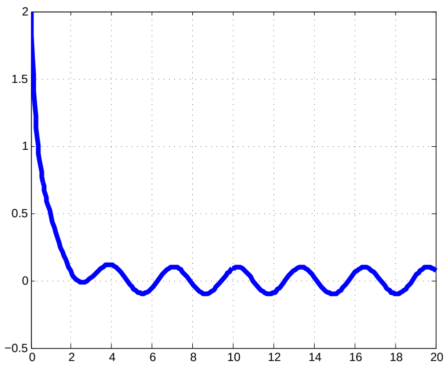

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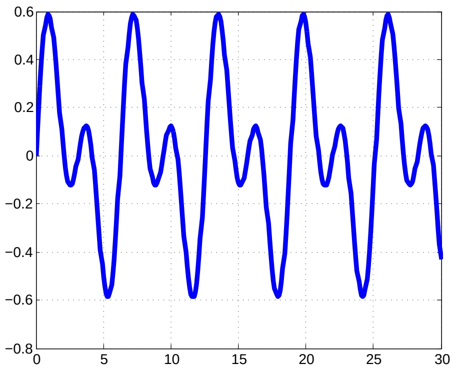

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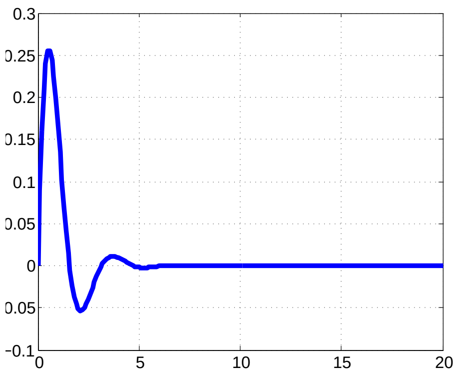

(g)

Differential Equation	Graph
$y'' + 4y = t$	b
$y'' + 4y = \cos(2t)$	d
$y'' + 4y = \sin(t)$	f
$y'' + 5y' + 4y = 0$	c
$y'' + 5y' + 4y = \cos(2t)$	e
$y'' + 2y' + 5y = 0$	g

Submitted by Name: ______________________________ *on November 15, 2019.*