Name: ____________________________________

Mathematics 307 L
University of Washington

October 16, 2019

MIDTERM 1

Here are the rules:

- This exam is closed book. No note sheets, calculators, or electronic devices are allowed.
- In order to receive credit, you must **show all of your work**; to obtain full credit, you must provide mathematical justifications. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Give numerical answers in exact form (for example $\ln(\frac{\pi}{3})$ or $5\sqrt{3}$ or $e^{2.5}$).
- Simplify $e^{a\ln(x)} = x^a$ for $x > 0$.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 6 pages, plus a cover sheet. Please make sure that your exam is complete.
- You have 50 minutes to complete the exam.
- HAVE FUN!

Problem	Possible	Score
1	10	
2	8	
3	10	
4	11	
5	16	
Total	55	

Good Luck!

Problem 1. (10 points) Consider the initial value problem,

$$\frac{dy}{dt} + \frac{y}{10+2t} = 3, \qquad y(0) = 0.$$

(a) (2 points) Circle your answer

(i) Is this a *linear* differential equation? **[YES]** **NO**

(ii) Is this a *separable* differential equation? **YES** **[NO]**

(b) (8 points) Solve the initial value problem.

Solution. Integrating factor solves

$$\frac{d\mu}{dt} = \frac{\mu}{10+2t} \;\Rightarrow\; \frac{d\mu}{\mu} = \frac{dt}{10+2t} \Rightarrow \ln(\mu) = \frac{1}{2}\ln(10+2t)$$

so

$$\mu = e^{\frac{1}{2}\ln(10+2t)} = (10+2t)^{\frac{1}{2}}$$

Equation becomes

$$(10+2t)^{\frac{1}{2}}\frac{dy}{dt} + (10+2t)^{-\frac{1}{2}}y = 3\,(10+2t)^{\frac{1}{2}}$$

which we rewrite as

$$\frac{d}{dt}\Big((10+2t)^{\frac{1}{2}}y\Big) = 3\,(10+2t)^{\frac{1}{2}}$$

Integrate to get

$$(10+2t)^{\frac{1}{2}}y = (10+2t)^{\frac{3}{2}} + C$$

Set $t=0$ and $y=0$ to solve

$$C = -10^{\frac{3}{2}}$$

Solve for y to get

$$\boxed{y = (10+t) - 10^{\frac{3}{2}}(10+2t)^{-\frac{1}{2}}}$$

Problem 2. (8 points) You borrow \$100 from your credit company at 5% annual interest, compounded continuously. You repay the loan at a continuous rate of \$10 per year. How many years does it take to pay off the loan?

Solution. Amount of the loan satisfies

$$\frac{dP}{dt} = .05\,P - 10 = \frac{1}{20}(P - 200)$$

Solve this most easily by separation of variables

$$\frac{dP}{P - 200} = \frac{1}{20}\,dt$$

$$\ln|P - 200| = \frac{t}{20} + C$$

$$P - 200 = \pm e^{C}\,e^{\frac{t}{20}}$$

Setting $t = 0$ and $P = 100$ leads to setting $\pm e^{C} = -100$, and then

$$P = 200 - 100\,e^{\frac{t}{20}}$$

Loan is paid off when $P = 0$ which happens when

$$e^{\frac{t}{20}} = 2 \quad \Rightarrow \quad \boxed{t = 20\ln(2)}.$$

Problem 3. (10 points) Consider the initial value problem

$$t\,\frac{dy}{dt} = \frac{-1}{y+1}, \qquad y(1) = -4.$$

(a) (8 points) Solve the initial value problem. Give an explicit formula for y.

Solution. Use separation of variables

$$(y+1)\,dy = \frac{-dt}{t}$$

$$\frac{1}{2}(y+1)^2 = -\ln|t| + C$$

Set $y = -4$ and $t = 1$ and use $\ln(1) = 0$ to get $C = \frac{9}{2}$.
Multiply by 2 and take square root to get

$$y + 1 = \pm\sqrt{9 - 2\ln|t|}$$

Since $y(1) = -4$, we must take the negative root, and then obtain

$$\boxed{y = -1 - \sqrt{9 - 2\ln|t|}}$$

Remark: Since the initial condition is at $t = 1$ you can write $\ln(t)$ instead of $\ln|t|$ everywhere above. This is a subtle point, since the formula above is a solution for $t < 0$ as well. We won't worry about this here; either answer is fine.

(b) (2 points) Find the largest interval of t containing $t = 1$ on which your answer $y(t)$ is defined.

Solution. The argument inside the root is defined for $t \neq 0$. The square root is defined only for positive arguments, so we also need

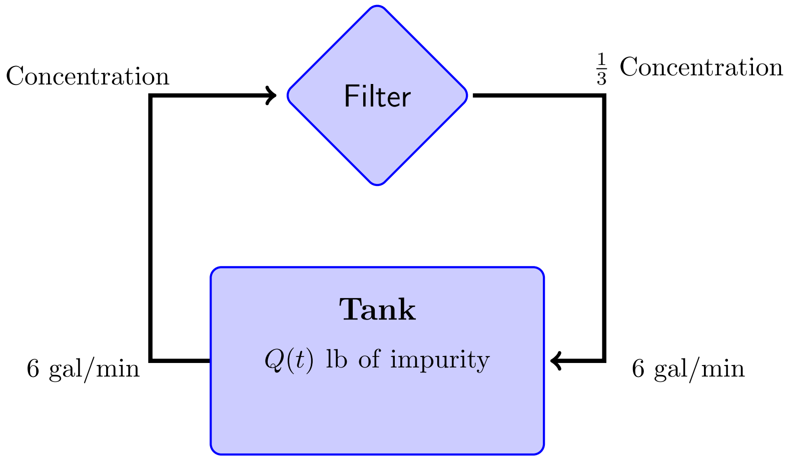
$$2\ln|t| \leq 9 \quad \Rightarrow \quad |t| \leq e^{\frac{9}{2}}.$$

The largest interval containing the point $t = 1$ that satisfies these conditions is

$$\boxed{0 < t \leq \tfrac{9}{2}}$$

but $\boxed{0 < t < \tfrac{9}{2}}$ is also perfectly fine as an answer.

Problem 4. (11 points) A tank holding water that contains an impurity Q is attached to a recirculating filter, as pictured below. The liquid passes through the filter at the rate of 6 gal/min. The filter removes 2/3 of the amount of Q that passes through it, and lets the remaining 1/3 go back into the tank.



(a) (5 points) The tank contains 10 gallons of water. Initially there are 3 pounds of Q dissolved in the water. Pose a differential equation with initial value for the amount $Q(t)$ in the tank at time t.

Solution. Amount of Q passing through the filter per minute is $6 \times \frac{Q}{V} = \frac{6}{10}Q$. The filter removes $\frac{2}{3}$ of this from the system, or $\frac{2}{3} \cdot \frac{6}{10} Q = \frac{4}{10} Q$ per minute,

$$\boxed{\frac{dQ}{dt} = -\frac{4}{10} Q\,, \qquad Q(0) = 3.}$$

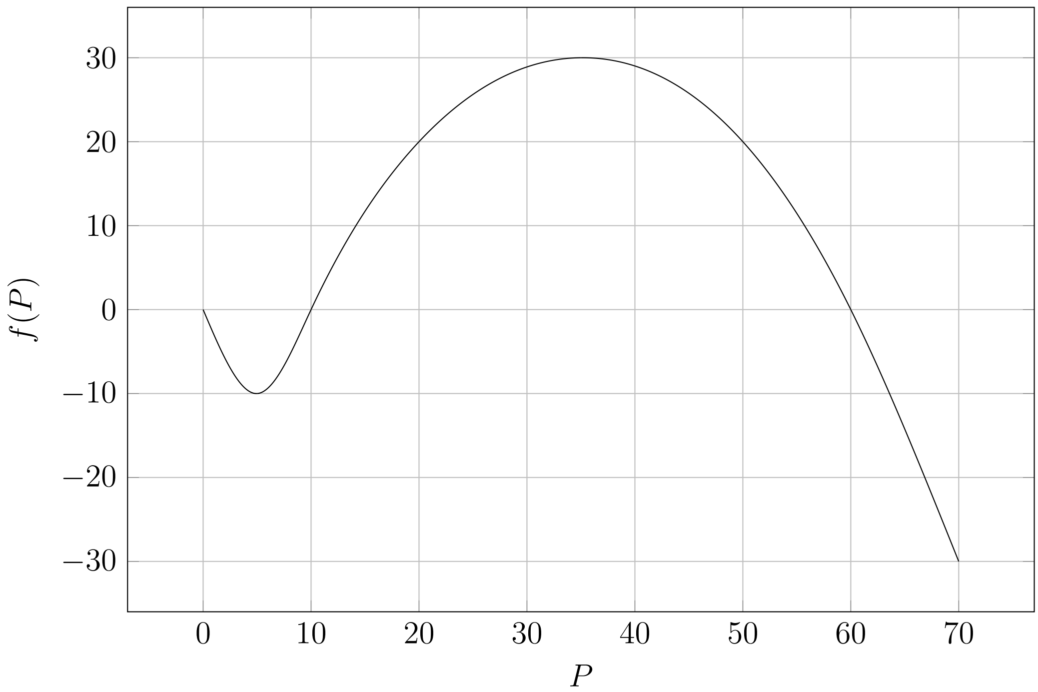
(b) (6 points) Now suppose that the tank initially contains 3 pounds of Q dissolved in 10 gallons of water. Water containing 1 pound of Q per gallon is added to the tank at the rate of 2 gallons per minute. The volume of water in the tank therefore increases. The filter continues to operate as above at 6 gallons per minute, removing 2/3 of the amount of Q passing through it. Pose a differential equation with initial value for the amount $Q(t)$ in the tank at time t.

Solution. Now the volume is $10 + 2t$, and Q is added at the rate 2 lb/min. Q is removed by the filter at the rate $\frac{2}{3} \cdot 6\,\frac{Q}{V} = \frac{4Q}{10 + 2t}$, so now

$$\boxed{\frac{dQ}{dt} = 2 - \frac{4}{10 + 2t} Q\,, \qquad Q(0) = 3.}$$

No need to solve this, but notice that it is very similar to problem 1 ☺.

Problem 5. (16 points) The population P of goldfish in a pond is observed to satisfy an equation $P' = f(P)$, where $f(P)$ is plotted here as a function of P.



(a) (3 points) Determine the equilibrium solutions and classify each one as stable or unstable.

Solution. Equilibrium are where $f(P) = 0$, or $P = 0$, 10, 60. The value is stable if $f(P)$ decreases there, unstable if $f(P)$ increases.

$$\boxed{\text{stable: } P = 0,\ 60 \qquad \text{unstable: } P = 10}$$

(b) (2 points) If the initial population satisfies $P(0) = 20$, what will happen to the population as t increases to ∞?

Solution. $f(P)$ is strictly positive for $10 < P < 60$, so P will increase whenever it lies in this range. It cannot go past 60 since $f(P) \leq 0$ when $P \geq 60$.

So if $P(0) = 20$, then $\boxed{\lim_{t\to\infty} P(t) = 60}$.

(c) (2 points) What is the minimum sustainable population? That is, what is the smallest positive value of $P(0)$ for which the population does not eventually go to 0?

Solution. If $0 < P < 10$, then $f(P)$ is negative, so a solution $P(t)$ will decrease towards 0 if $0 < P(0) < 10$ (and will stay at 0 if $P(0) = 0$). If $P(0) = 10$ then $P(t) = 10$ for all t. So the minimal sustainable population is $\boxed{P(0) = 10}$.

These exciting solutions continue on the next page!

For the problems on this page, we consider the above population of goldfish, and catch them at a rate of R fish per year.

(d) (3 points) If the population satisfies $P(0) = 50$, what is the maximum rate R at which you can catch fish without causing the population to eventually go to 0?

Solution. $\boxed{R = 30}$. The growth rate of the fish is at most 30 per year, which happens when $P \approx 35$. If you catch at more than 30 per year, then they can never grow fast enough to replace the catch. If you start at $P(0) = 50$ and catch at exactly 30 per year, then the population will decrease until it hits the semi-stable equilibrium at $P \approx 35$.

(e) (4 points) You catch fish at the rate of $R = 20$ fish per year. Write the new equation for P', determine the equilibrium solutions, and classify each one as stable or unstable.

Solution. The new equation is $\boxed{P' = f(P) - 20}$. This has equilibriums where $f(P) = 20$, so by the plot the equilibrium points are $P = 20$, 50. An equlibrium is stable if $f(P)$ is decreasing, unstable if $f(P)$ is increasing, so

$$\boxed{\text{unstable: } P = 20\,, \qquad \text{stable: } P = 50\,.}$$

(f) (2 points) You catch fish at the rate of $R = 20$ per year. If $P(0) = 30$, what will happen to the population as t increases to ∞?

Solution. $f(P) - 20$ is positive for $20 < P < 50$, so P will increase whenever it lies in this range. Since we start with $P(0) = 30$, P will keep increasing as long as it is less than 50. So $\boxed{\lim_{t\to\infty} P(t) = 50}$.

Submitted by Name: __________________________________ *on October 16, 2019.*