Name: Solutions
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WORKSHEET: LAPLACE TRANSFORM

Problem 1. Find the Laplace transform for

$$\text{(a). } f(t) = \begin{cases} 1 & 0 \le t < 4 \\ 3 & 4 \le t < 5 \\ 0 & 5 \le t < \infty \end{cases} \qquad \text{(b). } f(t) = \begin{cases} t & 0 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases}$$

$$\text{(c). } f(t) = \begin{cases} t^2 & 0 \le t < 1 \\ 0 & 1 \le t < \infty \end{cases} \qquad \text{(d). } f(t) = \begin{cases} \sin(t) & 0 \le t < \pi \\ \cos(t) & \pi \le t < \infty \end{cases}$$

Part (a) Solutions: First rewrite $f(t)$ in terms of heavyside functions:

$$f(t) = 1 + (3-1)u_4(t) + (0-3)u_5(t) = 1 + 2u_4(t) - 3u_5(t).$$

Using the linearity of the Laplace transform and the table:

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \underbrace{\mathcal{L}\{1\}}_{\#1} + 2\underbrace{\mathcal{L}\{u_4(t)\}}_{\#12 \text{ w/ } c=4} - 3\underbrace{\mathcal{L}\{u_5(t)\}}_{\#12 \text{ w/ } c=5} \\ &= \frac{1}{s} + \frac{2e^{-4s}}{s} - \frac{3e^{-5s}}{s} \end{aligned}$$

Part (b) Solutions: First rewrite $f(t)$ in terms of heavyside functions:

$$f(t) = t - tu_2(t).$$

Using linearity, $\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{tu_2(t)\}$. For $\mathcal{L}\{tu_2(t)\}$, we want it to look like #13 with $c = 2$. Thefore $t = f(t-2)$. In order to take the Laplace transform, we must find $f(v)$ so let $v = t-2$ or $v + 2 = t$. Then $f(v) = v + 2$ and

$$\mathcal{L}\{f(v)\} = \underbrace{\mathcal{L}\{v\}}_{\#3 \text{ w/ } n=1} + 2\underbrace{\mathcal{L}\{1\}}_{\#1} = \frac{1}{s^2} + \frac{2}{s}.$$

Using # 13,

$$\mathcal{L}\{tu_2(t)\} = \mathcal{L}\{v+2\}e^{-2t} = \left(\frac{1}{s^2} + \frac{2}{s}\right)e^{-2s}.$$

Putting everything together

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{t\} - \mathcal{L}\{tu_2(t)\} = \frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

Part (c) Solutions: Rewrite $f(t)$ in terms of heavyside functions:

$$f(t) = t^2 + (0 - t^2)u_1(t) = t^2 - t^2u_1(t).$$

The linearity of Laplace transform gives that $\mathcal{L}\{f(t)\} = \mathcal{L}\{t^2\} - \mathcal{L}\{t^2 u_1(t)\}$. For $\mathcal{L}\{t^2 u_1(t)\}$, we need to use #13. Let $t^2 = f(t-1)$. Set $v = t-1$ ($v+1=t$) so $f(v) = (v+1)^2 = v^2+2v+1$. Then

$$\mathcal{L}\{f(v)\} = \underbrace{\mathcal{L}\{v^2\}}_{\#3 \text{ w/ } n=2} + 2\underbrace{\mathcal{L}\{v\}}_{\#3 \text{ w/ } n=1} + \underbrace{\mathcal{L}\{1\}}_{\#1} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}.$$

Hence,

$$\mathcal{L}\{t^2 u_1(t)\} = \mathcal{L}\{f(v)\}e^{-s} = \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)e^{-s}.$$

Therefore,

$$\mathcal{L}\{f(t)\} = \underbrace{\mathcal{L}\{t^2\}}_{\#3 \text{ w/ } n=2} - \mathcal{L}\{t^2 u_1(t)\} = \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right)$$

Part (d) Solutions: Rewrite $f(t)$ in terms of heavyside functions:

$$f(t) = \sin(t) + \big(\cos(t) - \sin(t)\big)u_\pi(t).$$

Using linearity of Laplace transform, $\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(t)\} + \mathcal{L}\{\big(\cos(t) - \sin(t)\big)u_\pi(t)\}$. For $\mathcal{L}\{\big(\cos(t) - \sin(t)\big)u_\pi(t)\}$, we need $\cos(t) - \sin(t) = f(t-\pi)$ and to find $f(t)$. Set $v = t - \pi$ so $v + \pi = t$. Then $\cos(v+\pi) - \sin(v+\pi) = f(v)$. Unfortunately, one does not know $\mathcal{L}\{\cos(v+\pi)\}$ or $\mathcal{L}\{\sin(v+\pi)\}$; however by trig. identities $\cos(v+\pi) = -\cos(v)$ and $\sin(v+\pi) = -\sin(v)$. Therefore, $f(v) = \sin(v) - \cos(v)$. It follows that

$$\mathcal{L}\{f(v)\} = \underbrace{\mathcal{L}\{\sin(v)\}}_{\#5 \text{ w/ } a=1} - \underbrace{\mathcal{L}\{\cos(v)\}}_{\#6 \text{ w/ } a=1} = \frac{1}{s^2+1} - \frac{s}{s^2+1}.$$

Hence,

$$\mathcal{L}\{\big(\cos(t) - \sin(t)\big)u_\pi(t)\} = \mathcal{L}\{f(v)\}e^{-\pi s} = e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{s}{s^2+1}\right).$$

Consequently,

$$\mathcal{L}\{f(t)\} = \underbrace{\mathcal{L}\{\sin(t)\}}_{\#5 \text{ w/ } a=1} - \mathcal{L}\{\big(\cos(t) - \sin(t)\big)u_\pi(t)\} = \frac{1}{s^2+1} - e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{s}{s^2+1}\right)$$

Problem 2. Find the inverse Laplace transform for

$$(a).\ F(s) = \frac{e^{-3s}}{s^2 - 2s - 3} \qquad (b).\ F(s) = \frac{3!}{(s-2)^4}$$

$$(c).\ F(s) = \frac{2(s-1)e^{-2s}}{s^2 - 2s + 2} \qquad (d).\ F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$$

Part (a) Solutions: This looks like #13 on the Table. Let $H(s) = \frac{1}{s^2-2s-3}$. Then

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2 - 2s - 3}\right\} = \mathcal{L}^{-1}\{e^{-3s}H(s)\} = u_3(t)h(t-3).$$

It suffices to find $h(t) = \mathcal{L}^{-1}\{H(s)\}$. Observe by partial fractions

$$H(s) = \frac{1}{s^2 - 2s - 3} = \frac{1}{4}\left(\underbrace{\frac{1}{s-3}}_{\#2 \text{ w/ } a=3}\right) - \frac{1}{4}\left(\underbrace{\frac{1}{s+1}}_{\#2 \text{ w/ } a=-1}\right).$$

Hence,

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}.$$

Consequently,

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = u_3(t)h(t-3) = u_3(t)\left(\frac{1}{4}e^{3(t-3)} - \frac{1}{4}e^{-(t-3)}\right)}$$

Part (b) Solutions: Let $G(s-2) = F(s) = \frac{3!}{(s-2)^4}$. Set $w = s-2$ so $w+2 = s$ and $G(w) = \frac{3!}{w^4}$. The idea is to use #14 on the table with $G(s-2)$. Observe that $\mathcal{L}^{-1}\{G(w)\} = t^3$ by # 3. By #14,

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{G(s-2)\} = e^{2t}t^3.}$$

Part (c) Solutions: The idea is to use # 13. Let

$$H(s) = \frac{2(s-1)}{s^2 - 2s + 2} = \frac{2(s-1)}{(s-1)^2 + 1}$$

by completing the square. By #10 with $a = 1$ and $b = 1$, $h(t) = \mathcal{L}^{-1}\{H(s)\} = 2e^t\cos(t)$.
By #13,

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = u_2(t)h(t-2) = 2u_2(t)e^{t-2}\cos(t-2).}$$

Part (d) Solution: The idea is to apply #12 repeatedly:

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = u_1(t) + u_2(t) - u_3(t) - u_4(t).}$$

Problem 3. Find the solution of each of the following initial value problems

(a) $y'' + 2y' + y = 2(t-3)u_3(t); \qquad y(0) = 2, y'(0) = 1$

(b) $y'' + 4y = \begin{cases} 1 & 0 \le t < 4 \\ 0 & 4 \le t < \infty \end{cases} \qquad y(0) = 3, y'(0) = -2$

(c) $y'' - 2y' + y = \begin{cases} 0 & 0 \le t < 1 \\ t & 1 \le t < 2 \\ 0 & 2 \le t < \infty \end{cases} \qquad y(0) = 0, y'(0) = 1$

Part (a) Solutions: Take the Laplace transform of both sides:

$$s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) = \mathcal{L}\{2(t-3)u_3(t)\}.$$

Observe that we want to use # 13 and so need $f(t-3) = 2(t-3)$. Setting $v = t-3$ (i.e. $v+3 = t$), $f(v) = 2v$ with $\mathcal{L}\{v\} = \frac{2}{v^2}$ (# 3 with $n = 1$). Then $\mathcal{L}\{2(t-3)u_3(t)\} = \frac{2e^{-3s}}{s^2}$. Using this result and the initial conditions,

$$(s^2 + 2s + 1)Y(s) - 2s - 5 = \frac{2e^{-3s}}{s^2}.$$

or equivalently,

$$\begin{aligned} Y(s) &= \frac{2e^{-3s}}{s^2(s^2+2s+1)} + \frac{2s+5}{s^2+2s+1} \\ \text{partial fractions} \qquad &= e^{-3s}\left(\frac{2}{s^2} + \frac{4}{s+1} + \frac{2}{(s+1)^2} - \frac{4}{s}\right) + 2\left(\frac{s+\frac{5}{2}}{(s+1)^2}\right) \\ &= e^{-3s}\left(\frac{2}{s^2} + \frac{4}{s+1} + \frac{2}{(s+1)^2} - \frac{4}{s}\right) + 2\left(\frac{s+1}{(s+1)^2} + \frac{\frac{3}{2}}{(s+1)^2}\right). \end{aligned}$$

For the first term, it looks like # 13, so

$$G(s) = \underbrace{\frac{2}{s^2}}_{\#3 \text{ w/ } n=1} + \underbrace{\frac{4}{s+1}}_{\#14 \text{ and } \#1} + \underbrace{\frac{2}{(s+1)^2}}_{\#14 \text{ and } \#3} - \underbrace{\frac{4}{s}}_{\#1}$$

Hence,

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = 2t + 4e^{-t} + 2e^{-t}t - 4.$$

and so

$$\mathcal{L}^{-1}\{e^{-3s}G(s)\} = u_3(t)g(t-3) = u_3(t)\big(2(t-3) + 4e^{-(t-3)} + 2e^{-(t-3)}(t-3) - 4\big).$$

On the other hand, the second term is precisely

$$H(s) = 2\underbrace{\frac{s+1}{(s+1)^2}}_{\#2 \text{ w/ } a=-1} + \underbrace{\frac{3}{(s+1)^2}}_{\#14 \text{ and } \#3},$$

so

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = 2e^{-t} + 3te^{-t}.$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_3(t)g(t-3) = u_3(t)\big(2(t-3)+4e^{-(t-3)}+2e^{-(t-3)}(t-3)-4\big)+2e^{-t}+3te^{-t}.$$

Part (b) Solutions: Write $f(t)$ in terms of heavyside functions: $f(t) = 1+(0-1)u_4(t) = 1-u_4(t)$. Observe that the Laplace transform of $f(t)$ is

$$\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{e^{-4s}}{s}.$$

Taking Laplace transform of the differential equation and plugging in initial conditions:

$$(s^2+4)Y(s) - 3s + 2 = \frac{1}{s} - \frac{e^{-4s}}{s}$$

$$\begin{aligned} Y(s) &= \frac{1}{s(s^2+4)} - \frac{e^{-4s}}{s(s^2+4)} + \frac{3s-2}{s^2+4} \\ \text{partial fractions} \quad &= \frac{1}{4}\cdot\frac{1}{s} - \frac{s}{4(s^2+4)} - e^{-4s}\left(\frac{1}{4}\cdot\frac{1}{s} - \frac{s}{4(s^2+4)}\right) + 3\left(\frac{s}{s^2+4}\right) - \frac{2}{s^2+4}. \end{aligned}$$

Take the inverse Laplace transform: $\mathcal{L}^{-1}\left\{\frac{1}{4s} - \frac{s}{4(s^2+4)}\right\} = \frac{1}{4} - \frac{1}{4}\cos(2t).$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \tfrac{1}{4} - \tfrac{1}{4}\cos(2t) - u_4(t)\left(\tfrac{1}{4} - \tfrac{1}{4}\cos(2(t-4))\right) + 3\cos(2t) - \sin(2t).$$

Part (c) Solutions: Write $f(t)$ in terms of heavyside functions:

$$f(t) = tu_1(t) + (0-t)u_2(t) = tu_1(t) - tu_2(t).$$

Observe that we want $t = f_1(t-1)$ and $t = f_2(t-2)$. Setting $v = t-1$ and $w = t-2$, we have that $f_1(v) = v+1$ and $f_2(w) = v+2$. It follows that

$$\mathcal{L}\{f_1(v)\} = \frac{1}{s^2} + \frac{1}{s} \text{ and } \mathcal{L}\{f_2(t)\} = \frac{1}{s^2} + \frac{2}{s}.$$

Then by # 13,

$$\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right).$$

Taking Laplace transformations of the differential equation and plugging in initial conditions

$$(s^2 - 2s + 1)Y(s) - 1 = e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right).$$

$$\begin{aligned} \text{Therefore,} \quad Y(s) &= e^{-s}\left(\frac{1+s}{s^2(s-1)^2}\right) - e^{-2s}\left(\frac{1+2s}{s^2(s-1)^2}\right) + \frac{1}{(s-1)^2} \\ \text{Partial fractions} \quad &= e^{-s}\left(\frac{1}{s} + \frac{3}{s} - \frac{3}{s-1} + \frac{2}{(s-1)^2}\right) - e^{-2s}\left(\frac{1}{s^2} + \frac{4}{s} - \frac{4}{s-1} + \frac{3}{(s-1)^2}\right) + \frac{1}{(s-1)^2}. \end{aligned}$$

Most are directly off the table, but for $\frac{1}{(s-1)^2}$. To find the inverse Laplace transform, use #14:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^t\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = te^t.$$

Using #13,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = u_1(t)\big(t-1+3-3e^{t-1}+2(t-1)e^{t-1}\big) - u_2(t)\big(t-2+4-4e^{t-2}+3(t-2)e^{t-2}\big) + te^t$$