Lp Bounds for Spectral Clusters on Compact Manifolds with Boundary

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(M, g) = compact 2-d Riemannian manifold

- Δ_{g} = Laplacian (Dirichlet or Neumann if $\partial M \neq \emptyset$)
  Eigenbasis: $-\Delta_{\mathrm{g}}\phi_j = -\lambda_j^2\,\phi_j$ (λ_j = frequency)
- Spectral Cluster, frequency λ :

$$f = \sum_{\lambda_j \in [\lambda, \lambda+1]} c_j\,\phi_j$$

- Goal: find sharp powers $\delta(p)$ such that

$$\frac{\|f\|_{L^p(M)}}{\|f\|_{L^2(M)}} \lesssim \lambda^{\delta(p)} \qquad (\, p \geq 2\,)$$

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Saturating examples on $(M, g) = $ 2-sphere S^2

Example 1: $f =$ highest weight spherical harmonic.

$$|f| \approx \left(1 + \lambda^{\frac{1}{2}} \sin(\phi)\right)^{-N}$$

$$\frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^{\frac{1}{2}(\frac{1}{2}-\frac{1}{p})}$$

Lower bound (critical region)

$$\delta(p) \geq \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{p}\right)$$

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Example 2: $f =$ zonal spherical harmonic, rotation invariant.

$$|f| \approx \big(1 + \lambda \cos(\phi)\big)^{-1/2}$$

$$\frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim \lambda^{\frac{1}{2}-\frac{2}{p}} = \lambda^{2(\frac{1}{2}-\frac{1}{p})-\frac{1}{2}}$$

Lower bound (sub-critical region)

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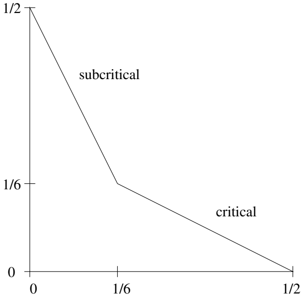
Theorem: Sogge [1988]

For compact n-dimensional manifold without boundary

$$\delta(p) = \begin{cases} \frac{n-1}{2}\left(\frac{1}{2} - \frac{1}{p}\right), & 2 \le p \le \frac{2(n+1)}{n-1} \\ n\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, & \frac{2(n+1)}{n-1} \le p \le \infty \end{cases}$$

$n = 2$



Grieser [1992]

Sogge's spectral cluster estimates fail on $D = \{\, |x| \leq 1 \,\} \subseteq \mathbb{R}^2$

Example: $f(x) = e^{in\theta}\, J_n(c_o r)\,, \qquad J_n(c_o) = 0\,.$

$f(x)$ concentrated in $\operatorname{dist}(x, \partial D) \lesssim n^{-\frac{2}{3}}\,\}$

$$\mathrm{Vol}(\text{“support”}(f)) \approx 1 \times n^{-\frac{2}{3}} \quad \Rightarrow \quad \frac{\|f\|_{L^p}}{\|f\|_{L^2}} \gtrsim n^{\frac{2}{3}(\frac{1}{2}-\frac{1}{p})}$$

Lower bound on manifolds with boundary

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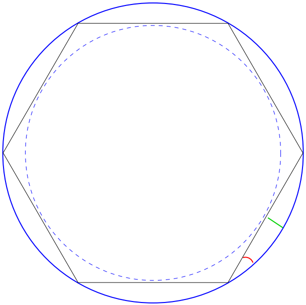
$$\delta(p) \;\geq\; \tfrac{2}{3}\big(\, \tfrac{1}{2} - \tfrac{1}{p} \,\big)$$

Multiple reflections / Gliding rays!

Nondispersive region: angular spread $\approx \lambda^{-\frac{1}{3}}$

physical spread $\approx \lambda^{-\frac{2}{3}}$



Smith-Sogge [1995]

∂M strictly concave $\Rightarrow$ spectral cluster estimates hold

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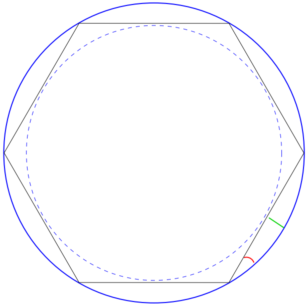

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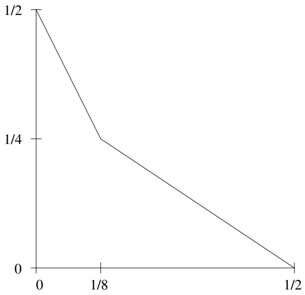
Smith-Sogge [2007]: M=2d manifold with boundary

Spectral cluster estimates hold with

$$\delta(p) = \begin{cases} \frac{2}{3}\left(\frac{1}{2} - \frac{1}{p}\right), & 6 \le p \le 8 \\ 2\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{1}{2}, & 8 \le p \le \infty \end{cases}$$



Key step in proof: Eliminate the boundary

Geodesic normal coordinates along ∂M : $M = \{x_2 \geq 0\}$

$$\mathrm{g} = d_{x_2}^2 + a_{11}(x_1, x_2)\, d_{x_1}^2 \,, \qquad \text{smooth on } x_2 \geq 0$$

Extend g across ∂M to be even in x_2

$$\tilde{\mathrm{g}} = d_{x_2}^2 + a_{11}(x_1, |x_2|)\, d_{x_1}^2$$

- Odd extension of Dirichlet eigenfunction: $\frac{x_2}{|x_2|}\,\phi_j(x_1, |x_2|)$ is eigenfunction for $\Delta_{\tilde{\mathrm{g}}}$
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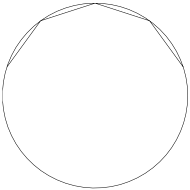
No more boundary / reflected geodesics:

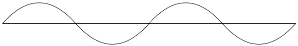
Disc: $r \leq 1$

$$\mathrm{g} = d_r^2 + \frac{1}{r^2} d_\theta^2$$

Normal coordinates: $x_2 = 1 - r$

$$\tilde{\mathrm{g}} = d_{x_2}^2 + \frac{1}{(1-|x_2|)^2} d_{x_1}^2$$





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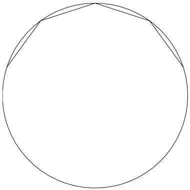

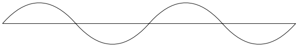

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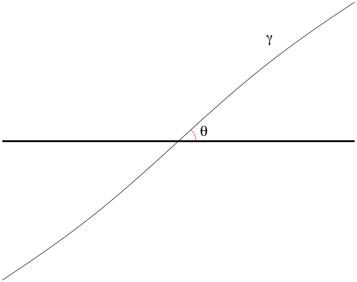
Metric $\tilde{g}$ is of special Lipschitz type:

$d_x^2 \tilde{g} \approx \delta(x_2)$ is integrable along non-tangential geodesics.



$$\frac{dx_2}{dt} \approx \theta \quad \text{on} \quad \gamma$$

$$\int d_x^2 \tilde{g}(\gamma(t))\, dt \approx \theta^{-1}$$

Dispersive type estimates hold for $\Box_{\mathrm{g}}$ if $d^2\mathrm{g} \in L^1_t L^\infty_x$

Consider time dependent metric $\mathrm{g}(t,x)$ on M

$$\partial_t^2 u(t,x) - \Delta_{\mathrm{g}} u(t,x) = 0$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Tataru [2002] : Strichartz estimates

If $\|\nabla^2_{t,x}\mathrm{g}\|_{L^1_t L^\infty_x} \leq 1$, then

$$\|u\|_{L^p_t L^q_x([-1,1]\times M)} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

for same p, q, s as smooth manifolds, Euclidean space.

In our case $\int d^2 \mathrm{g} \approx \theta^{-1}$

Rescaled metric $\mathrm{g}(\theta t, \theta x) \in L^1_t L^\infty_x$ norm 1:

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Squarefunction estimates

$$\| \cos(t\sqrt{-\Delta_g}\,)f(x)\|_{L^p_x L^2_t(M\times[-1,1])} \lesssim \|\langle D\rangle^{\delta(p)} f\|_{L^2(M)}\,, \quad p \geq 6$$

Squarefunction estimates $\Rightarrow$ spectral cluster bounds:

For spectral cluster f : $\cos(t\sqrt{-\Delta_g}\,)f(x) \approx \cos(t\lambda)f(x)$

$$\|f\|_{L^p(M)} \lesssim \| \cos(t\sqrt{-\Delta_g}\,)f(x)\|_{L^p_x L^2_t(M\times[-1,1])} \lesssim \lambda^{\delta(p)}\,\|f\|_{L^2(M)}$$

[Mockenhaupt-Seeger-Sogge (1993)] [S., $d^2 g \in L^1_t L^\infty_x$ (2006)]

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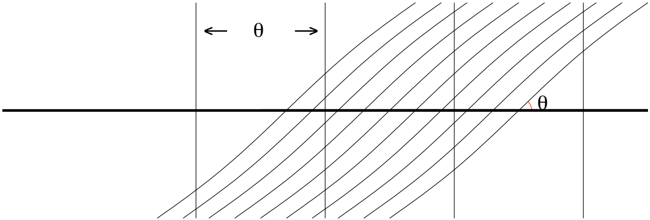
Phase-space localized spectral clusters:

If $\hat{f}(\xi_1, \xi_2)$ is localized to $\xi_2/\xi_1 \in [\theta, 2\theta]$, then we can prove "good" bounds on $\|f\|_{L^p}$ over slabs S of size θ in x_1 direction.



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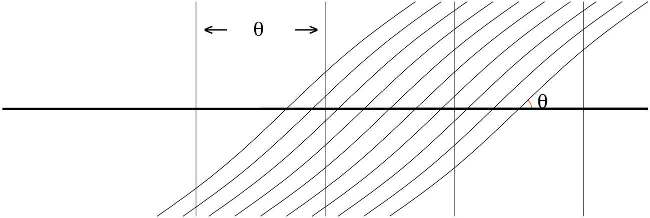

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For subcritical $p > 6$, gain from small angle localization

- If $\hat{f}_\theta$ is localized to a cone of angle θ, then

$$\|f_\theta\|_{L^p(S)} \lesssim \theta^{\frac{1}{2}-\frac{3}{p}} \lambda^{\delta(p)} \|f_\theta\|_{L^2(M)}$$

- Combined gain · loss for f_θ

$$\|f_\theta\|_{L^p(M)} \lesssim \theta^{\frac{1}{2}-\frac{4}{p}} \lambda^{\delta(p)} \|f_\theta\|_{L^2(M)}$$

- Sum over dyadic decomp in $\theta \leq 1$ yields

$$\|f\|_{L^p(M)} \lesssim \lambda^{\delta(p)} \|f\|_{L^2(M)}, \quad p \geq 8$$

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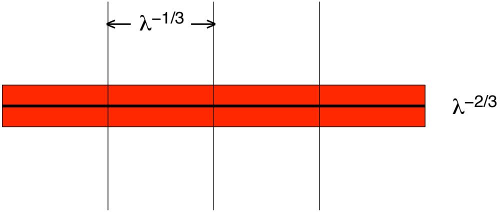
Gliding modes: $\theta = \lambda^{-1/3}$

- On slab S size $\lambda^{-1/3}$ in x_1:

$$\|f\|_{L^6(S)} \le \lambda^{1/6} \|f\|_{L^2(M)}$$

Sum over slabs:

$$\|f\|_{L^6(M)} \le \lambda^{1/6+1/18} \|f\|_{L^2(M)}$$



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$$\|f\|_{L^8(S)} \leq \lambda^{1/4}\lambda^{-1/24}\|f\|_{L^2(M)}$$

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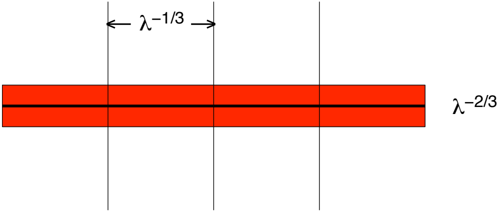

Higher dimensional results: $n \geq 3$

Smith-Sogge [2007]: $M = n$ dimensional manifold with boundary

No-loss square function / spectral cluster estimates hold with

$$\delta(p) = n(\tfrac{1}{2} - \tfrac{1}{p}) - \tfrac{1}{2} \qquad \begin{cases} 5 \leq p \leq \infty\,, & n = 3 \\ 4 \leq p \leq \infty\,, & n \geq 4 \end{cases}$$

Result non-optimal: ignores dispersion tangent to ∂M.