Strichartz Estimates for the Wave Equation on Manifolds with Boundary

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Wave equation on Riemannian manifold (M, g)

Cauchy problem: $$\partial_t^2 u(t,x) - \Delta_g u(t,x) = 0$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Strichartz estimates:

$$\|u\|_{L_t^p L_x^q((-T,T)\times M)} \lesssim \|f\|_{H^\gamma(M)} + \|g\|_{H^{\gamma-1}(M)}$$

Estimates hold on $\mathbb{R}^n$ or compact manifold without boundary if:

Scale invariance: $$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma$$

Admissibility: $$\frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}$$

Blair-S.-Sogge, 2008

(M, g)= compact manifold with boundary, Dirichlet or Neumann conditions at ∂M, then the Strichartz estimates hold if

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - \gamma \quad \text{for} \quad \begin{cases} \frac{3}{p} + \frac{n-1}{q} \leq \frac{n-1}{2}, & n \leq 4 \\ \frac{1}{p} + \frac{1}{q} \leq \frac{1}{2}, & n \geq 4 \end{cases}$$

- Scale invariant (no-loss), restriction on admissibility
- Estimates are *subcritical*: on $\mathbb{R}^n$ obtained from critical pair $(p, \tilde{q})$ followed by Sobolev embedding $W^{s,\tilde{q}} \subset L^q$
- Admissible range of p, q certainly not optimal, but includes important estimates.

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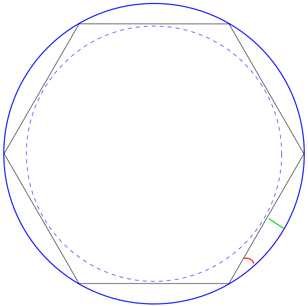
Multiple reflections / Gliding rays!

Nondispersive region: angular spread $\approx \lambda^{-\frac{1}{3}}$

physical spread $\approx \lambda^{-\frac{2}{3}}$



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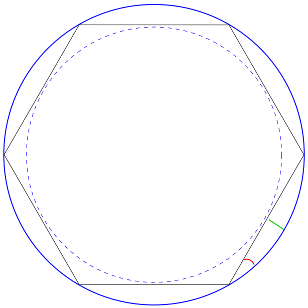

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Key step in proof: Eliminate the boundary

Geodesic normal coordinates along ∂M : $M = \{x_2 \geq 0\}$

$$\mathrm{g} = d_{x_2}^2 + a_{11}(x_1, x_2)\, d_{x_1}^2 \,, \qquad \text{smooth on } x_2 \geq 0$$

Extend g across ∂M to be even in x_2

$$\tilde{\mathrm{g}} = d_{x_2}^2 + a_{11}(x_1, |x_2|)\, d_{x_1}^2$$

- Odd extension of Dirichlet solution: $\frac{x_2}{|x_2|}\,\phi_j(x_1, |x_2|)$ is solution for $\partial_t^2 - \Delta_{\tilde{\mathrm{g}}}$
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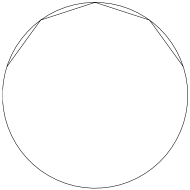
No more boundary / reflected geodesics:

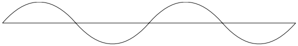
Disc: $r \leq 1$

$$\mathrm{g} = d_r^2 + \frac{1}{r^2} d_\theta^2$$

Normal coordinates: $x_2 = 1 - r$

$$\tilde{\mathrm{g}} = d_{x_2}^2 + \frac{1}{(1-|x_2|)^2} d_{x_1}^2$$





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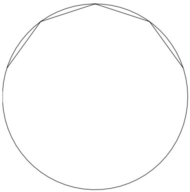
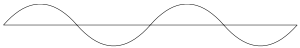
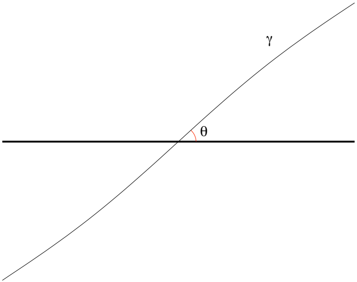




But metric is Lipschitz.

Metric $\tilde{g}$ is of special Lipschitz type:

$d_x^2 \tilde{g} \approx \delta(x_2)$ is integrable along non-tangential geodesics.



$$\frac{dx_2}{dt} \approx \theta \quad \text{on} \quad \gamma$$

$$\int d_x^2 \tilde{g}(\gamma(t))\, dt \approx \theta^{-1}$$

Tataru [2002] : Strichartz estimates

If $\|\nabla^2_{t,x}\mathrm{g}\|_{L^1_t L^\infty_x} \le \theta^{-1}$, then

$$\|u\|_{L^p_t L^q_x([-\theta,\theta]\times M)} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

for same p, q, s as smooth manifolds, Euclidean space.

Rescaled version of case $\theta = 1$:

If $\|\nabla^2_{t,x}\mathrm{g}\|_{L^1_t L^\infty_x} \le 1$, then

$$\|u\|_{L^p_t L^q_x([-1,1]\times M)} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}}$$

S-Sogge[2007]: Short time parametrix construction

Microlocalize data to angle θ: "Good" parametrix for time $|t| \leq \theta$

- Problem: add up $\|u_\theta\|_{L^p_t L^q_x}$ over time slices, lose $\theta^{-1/p}$

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Gain in bounds from small angle localization

$$K_{\lambda,\theta}(t,x) = \int_{\substack{\lambda\le|\xi|\le 2\lambda\\ |\xi_n|\le\theta\lambda}} e^{i\langle x,\xi\rangle - it|\xi|}\,d\xi$$

$$|K_{\lambda,\theta}(t,x)| \lesssim \lambda^n\theta\,(\,1+\lambda|t|\,)^{-\frac{n-2}{2}}\,(\,1+\lambda\theta^2|t|\,)^{-\frac{1}{2}}$$

- $\|K_{\lambda,\theta}(t,\cdot)*f\|_{L^p_tL^q_x} \lesssim \lambda^\gamma\theta^{\sigma(p,q)}\|f\|_{L^2}$
- No-loss Strichartz if $\sigma(p,q)\ge 1/p$

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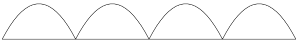
Friedlander model convex domain

Model domain: $y \geq 0$

$$\Delta_{\mathrm{g}} = \partial_y^2 + (1+y)\partial_x^2$$



Eigenfunctions: $e^{ix\xi} f(y)$

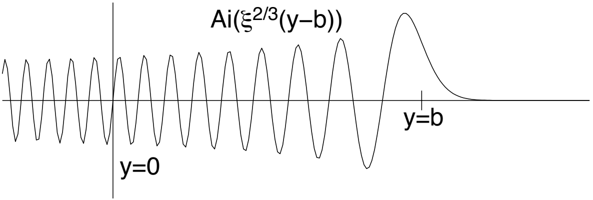
$$f''(y) = \xi^2 (1+y)\, f(y)$$

$$f(y) = Ai(\xi^{2/3}(y-b))$$

Eigenvalue: $\xi^2 (\, 1 + b\,)$

Solutions for $\partial_t^2 - \partial_y^2 - (1+y)\partial_x^2$

$e^{i(x-t\sqrt{1+b})\xi} Ai\big(\xi^{2/3}(y-b)\big)$ $\qquad$ x - velocity $= \sqrt{1+b}$



- Dirichlet: $-\xi^{2/3} b =$ zero of Airy function $\{-\omega_k\}_{k=0}^{\infty}$
- Fixed zero of Ai forces $b = \omega_k\, \xi^{-2/3} \Rightarrow$ dispersion.

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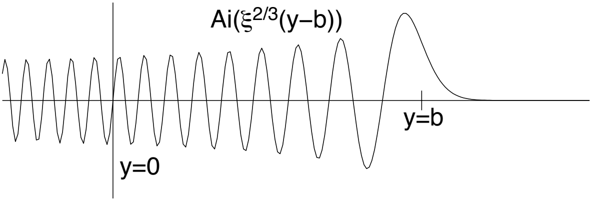
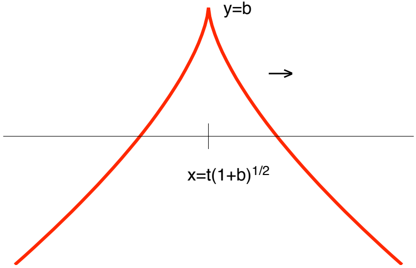


- Dirichlet: $-\xi^{2/3}b =$ zero of Airy function $\{-\omega_k\}_{k=0}^{\infty}$
- Fixed zero of Ai forces $b = \omega_k\,\xi^{-2/3} \Rightarrow$ dispersion.

Fix *b*, ignore boundary condition:

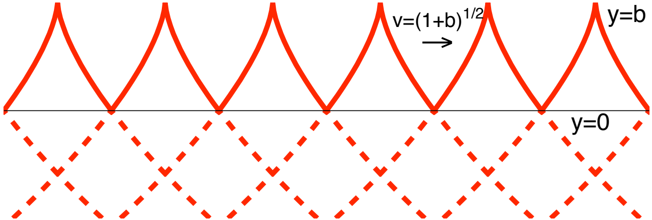
$$\int e^{i\xi(x-t\sqrt{1+b}\,)} Ai\big(\xi^{2/3}(y-b)\big)\,\xi^{-2/3}\,d\xi$$

$$= \int e^{i\xi(x-t\sqrt{1+b}\,)+i\eta(b-y)-i\frac{1}{3}\eta^3/\xi^2}\,d\xi\,d\eta$$



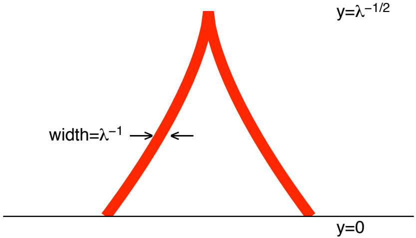
Boundary conditions: $\xi^{2/3}b = -\omega_k \; : \; \xi \approx k\pi \cdot \frac{3}{2} b^{-3/2}$

Poisson summation: periodize by $\frac{4}{3} b^{3/2}$



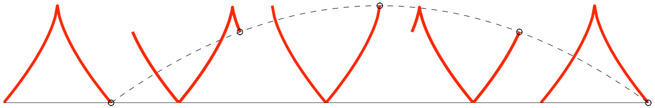
Ivanovici's example:

Single cusp localized to frequency $\xi \in [\lambda, 2\lambda]$ *with* $b = \lambda^{-\frac{1}{2}+\epsilon}$



$$|f_\lambda| \approx \langle \lambda^{2/3}(b-y)\rangle^{-1/4}\, \phi\big(\lambda(x \pm \tfrac{2}{3}(b-y)^{3/2})\big)$$

Ray optics tracking of cusp wavefront in domain:



$$\frac{\|u_\lambda(t,\cdot)\|_q}{\|u_\lambda(0,\cdot)\|_2} \approx \begin{cases} \lambda^{\frac{3}{4}(\frac{1}{2}-\frac{q}{4})} & q<4 \\ \lambda^{\frac{5}{3}(\frac{1}{2}-\frac{1}{q})-\frac{1}{24}} & q>4 \end{cases}$$

Ivanovici [2008] ($n=2$)

Strichartz estimates fail for $\frac{3}{p}+\frac{1}{q}>\frac{15}{24}$

Blair-S-Sogge [2008] ($n=2$)

Strichartz estimates hold for $\frac{3}{p}+\frac{1}{q}\leq\frac{1}{2}$

Energy critical wave equation for $n = 3$

$$\Box u(t,x) = -u^5(t,x)\,, \qquad u|_{\partial\Omega} = 0\,,$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Two key Strichartz estimates: $\Box u = 0$

$$\|u\|_{L^5_t L^{10}_x([0,T]\times\Omega)} + \|u\|_{L^4_t L^{12}_x([0,T]\times\Omega)} \lesssim \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$$

Energy critical wave equation for $n = 3$

- $L^5_t L^{10}_x$ contraction argument $\Rightarrow$ small data global existence

$$\|u^5\|_{L^1_t L^2_x} \le \|u\|^5_{L^5_t L^{10}_x}$$

- Grillakis-Shatah-Struwe [1992]: global existence for large data uses local L^6 decay:

$$\lim_{t \to t_0^-} \int_{|x-x_0| \le t_0 - t} |u|^6 \, dx = 0$$

$$\|u^5\|_{L^1_t L^2_x} \le \|u\|_{L^\infty_t L^6_x} \|u\|^4_{L^4_t L^{12}_x}$$

Scattering for $\Box u = -u^5$

Bahouri-Gérard-Shatah [1998-99]: $\Omega = \mathbb{R}^3$

There exists free solutions $\Box v^{\pm} = 0$ such that

$$\lim_{t \to \pm\infty} E(u - v_{\pm})(t) = 0$$

Key step: $$\lim_{t \to \pm\infty} \int_{\mathbb{R}^3} |u|^6(t, x)\, dx = 0$$

- Blair-S-Sogge [2008]: $\Omega =$ exterior to star-shaped obstacle.
- Strichartz estimates global in time on exterior to non-trapping obstacle by S-Sogge [2000]

Energy critical wave equation for $n = 4$

$$\Box u(t,x) = -u^3(t,x)\,, \qquad u|_{\partial\Omega} = 0\,,$$

$$u(0,x) = f(x)\,, \quad \partial_t u(0,x) = g(x)$$

Key Strichartz estimate: $\Box u = 0$

$$\|u\|_{L^3_t L^6_x([0,T]\times\Omega)} \lesssim \|f\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)}$$

- Contraction argument + energy conservation $\Rightarrow$ small data global existence, large data global existence for sub-critical powers.
- Large data global existence for critical power requires an estimate with $p < 3$, but $p = 3$ is endpoint for [B-S-S].