Optical Tomography in two dimensions

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Abstract

We consider in two dimensions, the inverse boundary problem of reconstructing the absorption and scattering coefficient of an inhomogeneous medium by probing it with diffuse light. The problem is modeled as an inverse boundary problem for the stationary linear Boltzmann equation. The information is encoded in the albedo operator. We show that we can recover the absorption and the scattering kernel from this information provided that the latter is small in an appropriate topology. We also give stability estimates.

Optical Tomography concerns with the determination of spatially-varying optical absorption and scattering properties of a medium by measuring the response of the medium to transmitted near-infrared light. This has been proposed recently as a possible diagnostic tool in medicine (see [A] for a review). A probabilistic approach to optical tomography is the so-called diffused tomography [SGKZ]. Other areas of applications are atmospheric remote sensing [Bi] , nuclear physics (see [MC] for a review), etc. We describe in more detail the mathematical problem below.

Let $X \subset \mathbf{R}^2$ be an open convex set with smooth boundary and consider the 2D stationary transport equation

$$-v(\theta) \cdot \nabla_x f - \sigma(x,\theta) f + \int_{S^1} k(x,\theta',\theta) f(x,\theta')\, d\theta' = 0, \tag{1}$$

where $x \in X$, $\theta, \theta' \in S^1$ and throughout this paper we will use the notation

$$v(\theta) = (\cos\theta, \sin\theta).$$

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Here $f(x,\theta)$ represents the density of a particle flow, $\sigma \geq 0$ is the absorption coefficient, while $k(x,\theta',\theta) \geq 0$ is the collision kernel. We assume that σ and k are in L^∞. Denote $\Gamma_\pm = \{(x,v) \in \partial X \times S^1;\ \pm n(x)\cdot v > 0\}$, where $n(x)$ is the outer normal to ∂X at $x \in \partial X$. Let us denote by f the solution (if exists) to (1) satisfying the boundary condition

$$f|_{\Gamma_-} = f_-, \tag{2}$$

where $f_- \in C_0^\infty(\Gamma_-)$. The albedo operator $\mathcal{A}$ mapping the incoming flux f_- on Γ_- to the outgoing one on Γ_+ is defined as follows

$$\mathcal{A} : f_- \longmapsto f|_{\Gamma_+}.$$

In this paper we work with k sufficiently small and under this assumption (1), (2) is always uniquely solvable with $f \in L^\infty(X \times S^1)$ and $\mathcal{A}f \in L^\infty(\Gamma_+)$.

The inverse problem we are interested in, is the problem of recovering (σ, k) from knowledge of the albedo operator $\mathcal{A}$. First, if $k = 0$, it is obvious that $\mathcal{A}$ does not determine uniquely σ since in this case all we get from the knowledge of $\mathcal{A}$ is $\int \sigma(x + t(\cos\theta, \sin\theta), \theta)\, dt$ for any $\theta \in S^1$ and any x and this is not enough to recover σ. For this reason in our main theorems we assume that σ depends on x only. In dimensions $n \geq 3$ this problem is formally overdetermined for the recovery of k and uniqueness under the assumption above is proved in [CS1], [CS2] by studying the singularities of the distributional kernel of $\mathcal{A}$ (see also [Bo]). Some of the earlier works where this problem is treated under more restrictive assumptions are [AB], [CZ], [L1], [L2], [MC], [PV]. Stability estimates under various assumptions are proved in [R2], [R3] (see also the references therein), [W].

In two dimensions, this inverse problem is formally determined for the recovery of k. Uniqueness and stability are proved for small k in the case when $k = k(x, \cos(\theta - \theta'))$ in [R1], and in the case $k = k(\theta, \theta')$ uniqueness for small k is proved in [T]. Note that in those cases the inverse problem is still formally overdetermined. The purpose of this paper is to prove a uniqueness results for general (small) $k(x,\theta',\theta)$ and a stability estimate.

The first main result in this paper is the following.

Theorem 1 *Define the class*

$$\mathcal{U}_{\Sigma,\varepsilon} = \{(\sigma(x), k(x,\theta',\theta))\,;\ \|\sigma\|_{L^\infty} \leq \Sigma,\ \|k\|_{L^\infty} \leq \varepsilon\}\,. \tag{3}$$

Then, for any $\Sigma > 0$ there exists $\varepsilon > 0$ such that a pair $(\sigma, k) \in \mathcal{U}_{\Sigma,\varepsilon}$ is uniquely determined by its albedo operator $\mathcal{A}$ in the class $\mathcal{U}_{\Sigma,\varepsilon}$.

The proof of Theorem 1 follows by a careful analysis of the singularities of the distribution kernel $\alpha(x,\theta,x',\theta')$ of $\mathcal{A}$ (see also [CS1], [CS2] for similar analysis in dimensions $n \geq 3$). In the two dimensional case we show that $\mathcal{A}$ admits the singular decomposition (see Lemma 1 and Proposition 1)

$$\alpha = \frac{\delta(\theta-\theta')\delta_{\{x'+\tau_+(x',\theta')v(\theta')\}}(x)}{n(x)\cdot v(\theta)} e^{-a(x',\theta')} + b(x,\theta,x',\theta'), \tag{4}$$

where $n(x)$ is the exterior normal to ∂X, the function $\tau_+(x',\theta')$ is the "exit time" defined in next section, and

$$a(x',\theta') = \int_0^{\tau_+(x',\theta')} \sigma(x'+tv(\theta'),\theta')\,dt, \quad \sin(\theta-\theta')b \in L^\infty. \tag{5}$$

In particular, knowing $\mathcal{A}$, we can uniquely determine a and b. Let we have two pairs (σ,k) and $(\tilde\sigma,\tilde k)$ with albedo operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. Set

$$\delta_1 = \|a-\tilde a\|_{H^1(\Gamma_-)}, \quad \delta_2 = \|(b-\tilde b)\sin(\theta-\theta')\|_{L^\infty(\Gamma_-\times\Gamma_+)}. \tag{6}$$

Our second result is the following stability estimate.

Theorem 2 *Let*

$$\mathcal{V}^s_{\Sigma,\varepsilon} = \left\{(\sigma(x),k(x,\theta',\theta)) \in H^s(X)\times C(X\times S^1\times S^1);\ \|\sigma\|_{H^s}\le\Sigma,\ \|k\|_{L^\infty}\le\varepsilon\right\}. \tag{7}$$

Then, for any $s>1$, $\Sigma>0$, there exists $\varepsilon>0$ such that for any $(\sigma,k)\in\mathcal{V}^s_{\Sigma,\varepsilon}$ and $(\tilde\sigma,\tilde k)\in\mathcal{V}^s_{\Sigma,\varepsilon}$ and $0<\mu<1-1/s$, there exists $C>0$ such that

$$\begin{aligned}\|\sigma-\tilde\sigma\|_{L^\infty} &\le C\delta_1^{1-1/s-\mu},\\ \|k-\tilde k\|_{L^\infty} &\le C\left(\delta_1^{1-1/s-\mu}+\delta_2\right).\end{aligned}$$

1 The uniqueness, proof of Theorem 1

Recall that σ and k are L^∞ in all variables. It is convenient to think later that σ and k are extended as 0 for $x\notin X$. Denote the left-hand side of (1) by Tf. Denote

$$T_1f = -v(\theta)\cdot\nabla_x f - \sigma f, \quad Kf = \int_{S^1} k(x,\theta',\theta)f(x,\theta')d\theta',$$

so that $T=T_1+K$. First, we reduce the boundary value problem

$$\left\{\begin{array}{rcl} Tf &=& 0 \quad \text{in } X\times S^1,\\ f|_{\Gamma_-} &=& f_-\in C_0^\infty(\Gamma_+)\end{array}\right. \tag{8}$$

to an integral equation by a standard argument. For $(x,\theta)\in\bar X\times S^1$, define $\tau_\pm(x,\theta)\ge 0$ by the condition $(x\pm\tau_\pm(x,\theta)v(\theta),\theta)\in\Gamma_\pm$. Introduce the operator M by

$$Mf = \int_0^\infty e^{-\int_0^t \sigma(x-sv(\theta),\theta)ds}(Kf)(x-tv(\theta),\theta)\,dt. \tag{9}$$

If we denote the solution to $T_1u=f$, $u|_{\Gamma_-}=0$ by

$$T_1^{-1}f = -\int_0^{\tau_-(x,\theta)} e^{-\int_0^t \sigma(x-sv(\theta),\theta)ds} f(x-tv(\theta),\theta)\,dt,$$

then $M = -T_1^{-1}K$. Problem (8) can be written as $(T_1 + K)f = 0$, $f|_{\Gamma_-} = f_-$. Applying T_1^{-1} to the first equation, we get that f solves

$$(I - M)f = Jf_-, \tag{10}$$

where

$$Jf_- = e^{-\int_0^{\tau_-(x,v)} \sigma(x-sv,v)ds} f_-(x - \tau_-(x,v)v, v).$$

Note that Jf_- solves the problem $T_1 Jf_- = 0$, $Jf_-|_{\Gamma_-} = f_-$. Then f is given by

$$f = (I - M)^{-1} Jf_-, \tag{11}$$

provided that $I - M$ is invertible in a suitable space. The definition (9) of M implies immediately that

$$\|Mf\|_{L^\infty(X\times S^1)} \leq C\|f\|_{L^\infty(X\times S^1)},$$

where $C = \mathrm{diam}(X)\|k\|_{L^\infty}$. Therefore, if $(\sigma, k) \in \mathcal{U}_{\Sigma,\varepsilon}$ and $\varepsilon < 1/\mathrm{diam}(X)$, then $I - M$ is invertible in $L^\infty(X \times S^1)$, and then the solution f to (8) is given by (11). By using Neumann series, it is not hard to see that the trace $(I - M)^{-1}f|_{\Gamma_+}$ is well defined in $L^\infty(\Gamma_+)$ for any $f \in L^\infty(X \times S^1)$. This proves in particular that $\mathcal{A}$ maps $C_0^\infty(\Gamma_-)$ into $L^\infty(\Gamma_+)$ under the smallness assumption on k above. The same arguments also show that $\mathcal{A}f_-$ can be defined for any $f_- \in L^\infty(\Gamma_-)$ as well but we will not need this since we work with the distribution kernel of $\mathcal{A}$. The mapping properties of $\mathcal{A}$ in L^1 spaces are discussed in [CS2].

We proceed with the study of the distribution kernel of $\mathcal{A}$. Define first the fundamental solution $\phi(x,\theta,x',\theta')$ of the boundary value problem (8) as follows. For $(x',v') \in \Gamma_-$, let $\phi(x,v,x',v')$ solve

$$\begin{cases} T\phi &= 0 \quad \text{in } X \times S^1, \\ \phi|_{\Gamma_-} &= |n(x')\cdot v(\theta')|^{-1}\delta_{x'}(x)\delta(\theta - \theta'), \end{cases} \tag{12}$$

where $\delta_{x'}(x) \in \mathcal{D}'(\partial X)$ is defined by $\int_{\partial X} \delta_{x'}(x)\varphi(x)\,d\mu(x) = \varphi(x')$, with $d\mu(x)$ the induced measure on ∂X. Then the albedo operator $\mathcal{A}$ has distribution kernel $\alpha(x,\theta,x',\theta') = \phi(x,\theta,x',\theta')|_{(x,\theta)\in\Gamma_+}$, $(x',\theta') \in \Gamma_-$, $(x,\theta) \in \Gamma_+$, i.e.,

$$(\mathcal{A}f_-)(x,v) = \int_{\Gamma_-} \alpha(x,\theta,x',\theta') f_-(x',\theta')\,d\xi(x',\theta'), \quad \forall f_- \in C_0^\infty(\Gamma_+),$$

where $d\xi(x',\theta') = |n(x')\cdot v(\theta')|d\xi(x',\theta')\,d\mu(x')\,d\theta'$ and $n(x)$ is the unit exterior normal to ∂X.

As in [CS2], we construct a singular expansion of ϕ as follows. Note that for any $f_- \in C_0^\infty(\Gamma_-)$, the outgoing flux $\mathcal{A}f_-$ is given by

$$\mathcal{A}f_- = Jf_-|_{\Gamma_+} + MJf_-|_{\Gamma_+} + (I - M)^{-1}M^2 Jf_-|_{\Gamma_+}.$$

Let

$$E(x,\theta,t) = e^{\mp\int_0^t \sigma(x+sv(\theta),\theta)ds}, \quad \pm t \geq 0$$

be the total absorption along the path $[x, x+tv(\theta)]$. Then

$$\phi = \phi_0 + \phi_1 + \phi_2,$$

where

$$\phi_0 = J\phi_-, \quad \phi_1 = MJ\phi_-, \quad \phi_2 = (I-M)^{-1}M^2 J\phi_-, \tag{13}$$

and $\phi_- = |n(x')\cdot v(\theta')|^{-1}\delta_{x'}(x)\delta(\theta-\theta')$ as in (12). As in [CS2],

$$\phi_0 = E(x,\theta,-\infty)\delta(\theta-\theta')\int_0^{\tau_+(x',v(\theta'))}\delta(x-x'-tv(\theta'))dt.$$

Next,

$$\phi_1 = \chi E(y,\theta',-\infty)\frac{k(y,\theta',\theta)}{|\sin(\theta-\theta')|}E(y,\theta,\infty), \tag{14}$$

where $y = y(x',\theta',x,v)$ is the point of intersection of the rays $(0,\infty)\ni s\mapsto x'+sv(\theta')$ and $(-\infty,0)\ni t\mapsto x+tv(\theta)$ and $\chi = \chi(x,\theta,x',\theta')$ equals 1, if those two rays intersect in $\bar{X}$, otherwise $\chi = 0$. Recall that X is convex.

To estimate ϕ_2 we need the following.

Lemma 1 *Let (σ,k) and $(\sigma,\tilde{k})$ be in L^∞. Let K, M and $\tilde{K}$, $\tilde{M}$ be related to k and $\tilde{k}$ (not necessarily non-negative), respectively. Then there exists $C>0$ depending on $\operatorname{diam}(X)$ only such that*

$$|(\tilde{M}M\phi_0)(x,\theta,x',\theta')| \le C\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\left(1+\log\frac{1}{\sin|\theta-\theta'|}\right)$$

almost everywhere on $X\times S^1\times\Gamma_-$, and also almost everywhere on $\Gamma_+\times\Gamma_-$.

Proof. Note first that for any f,

$$|Mf| \le \int_0^\infty\int_{S^1}|k(x-tv(\theta),\theta',\theta)f(x-tv(\theta),\theta')|\,d\theta'\,dt,$$

therefore

$$\begin{aligned}|(\tilde{K}Mf)(x,\theta)| &\le \int_{S^1}\int_0^\infty\int_{S^1}|\tilde{k}(x,\theta'',\theta)k(x-tv(\theta''),\theta'\theta'')f(x-tv(\theta''),\theta')|\,d\theta'\,dt\,d\theta'' \\ &= \int\int_{S^1}\frac{|\tilde{k}(x,\arg(x-y),\theta)k(y,\theta',\arg(x-y))|}{|x-y|}|f(y,\theta')|\,d\theta'\,dy.\end{aligned}$$

Here $\arg(z)$ is the polar angle of $z\in\mathbf{R}^2$. Setting $f=\phi_0$, we get

$$|(\tilde{K}M\phi_0)(x,\theta,x',\theta')| \le \int_{y\in l(x',\theta')}\frac{|\tilde{k}(x,\arg(x-y),\theta)k(y,\theta',\arg(x-y))|}{|x-y|}dl(y),$$

where $l(x',\theta')$ is the line through x' parallel to θ' and dl is the Euclidean measure on it. The following estimate

$$\int_{-A}^{A}\frac{ds}{\sqrt{\nu^2+s^2}} \le 2\left(1+\log\frac{A}{\nu}\right), \quad 0<\nu\le A.$$

is easy to prove by splitting the interval of integration into two parts: $|s| \leq 1$ and $1 \leq |s| \leq A$. Therefore,

$$|(\tilde{K}M\phi_0)(x,\theta,x',\theta')| \leq 2\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\left(1+\log\frac{d}{\operatorname{dist}\{x,l(x',\theta')\}}\right), \tag{15}$$

where $d = \operatorname{diam}(X)$. Applying T_1^{-1}, we get

$$\begin{aligned}|(\tilde{M}M\phi_0)(x,\theta,x',\theta')| &\leq 2\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\int_0^d\left(1+\log\frac{d}{\operatorname{dist}\{x-tv(\theta),l(x',\theta')\}}\right)dt\\ &\leq C\|\tilde{k}\|_{L^\infty}\|k\|_{L^\infty}\left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right),\end{aligned}$$

with $C = C(d)$. This proves the lemma. $\square$

For ϕ_2 we have therefore $\phi_2 = (I-M)^{-1}\phi_2^\#$ with $\phi_2^\# = M^2\phi_0$ and by Lemma 1,

$$0 \leq \phi_2^\#(x',\theta',x,\theta) \leq C\|k\|_{L^\infty}^2\left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right). \tag{16}$$

To estimate ϕ_2, write $\phi_2 = \phi_2^\# + (I-K)^{-1}K\phi_2^\#$. Since the singularity in the r.h.s. of (16) is integrable, we get that $K\phi_2^\# \in L^\infty$, therefore $\|\phi_2 - \phi_2^\#\|_{L^\infty} \leq C\|k\|_{L^\infty}^3$, with some constant $C > 0$, and in particular, $\phi_2 - \phi_2^\# \in L^\infty$. From now on, we will denote by C different positive constants. Therefore, ϕ_2 also satisfies (16). Observe that $0 \leq \phi_2$ because $(I-K)^{-1}$ preserves the non-negative functions.

We have therefore proved the following.

Proposition 1 *For $\varepsilon > 0$ small enough, the fundamental solution ϕ of (8) defined by (12) admits the representation $\phi = \phi_0 + \phi_1 + \phi_2$ with*

$$\begin{aligned}\phi_0 &= E(x,\theta,-\infty)\delta(\theta-\theta')\int_0^{\tau_+(x',v(\theta'))}\delta(x-x'-tv(\theta'))dt.\\ \phi_1 &= \chi E(y,\theta',-\infty)\frac{k(y,\theta',\theta)}{|\sin(\theta-\theta')|}E(y,\theta,\infty),\\ 0 \leq \phi_2 &\leq C\|k\|_{L^\infty}^2\left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right),\end{aligned}$$

where y, χ are as in (14) and $C = C(\operatorname{diam}(X))$.

Fix $\Sigma > 0$ and assume that we have two pairs (σ,k), $(\tilde{\sigma},\tilde{k})$ in $\mathcal{U}_{\Sigma,\epsilon}$ with the same albedo operator and σ, $\tilde{\sigma}$ depending on x only. Denote by ϕ_j and $\tilde{\phi}_j$, $j = 0,1,2$, the corresponding components of the fundamental solutions ϕ and $\tilde{\phi}$. Then

$$\phi_0 + \phi_1 + \phi_2 = \tilde{\phi}_0 + \tilde{\phi}_1 + \tilde{\phi}_2 \quad \text{for } (x,\theta) \in \Gamma_+. \tag{17}$$

As in [CS1], the most singular terms must agree, therefore,

$$\phi_0(x',\theta',x,\theta)=\tilde{\phi}_0(x',\theta',x,\theta) \quad \text{for } (x',\theta')\in\Gamma_-,\ (x,\theta)\in\Gamma_+. \tag{18}$$

Thus the X-ray transform of $\sigma(x)$ and $\tilde{\sigma}(x)$ coincide, therefore,

$$\sigma(x)=\tilde{\sigma}(x).$$

Next,

$$\chi E(y,\theta',-\infty)\frac{k(y,\theta',\theta)-\tilde{k}(y,\theta',\theta)}{|\sin(\theta-\theta')|}E(y,\theta,\infty)=\tilde{\phi}_2-\phi_2 \quad \text{on } \Gamma_-\times\Gamma_+,$$

where y, χ are as in (14). This, together with (17), (18), leads to the inequality

$$\chi|k(y,\theta',\theta)-\tilde{k}(y,\theta',\theta)|\leq C|\sin(\theta-\theta')||\tilde{\phi}_1-\phi_1|=C|\sin(\theta-\theta')||\tilde{\phi}_2-\phi_2| \quad \text{on } \Gamma_-\times\Gamma_+, \tag{19}$$

where $C=e^{2d\Sigma}$, $d=\mathrm{diam}(D)$. Our goal is to show that

$$\operatorname*{ess\,sup}_{\Gamma_-\times\Gamma_+}|\sin(\theta-\theta')||\tilde{\phi}_2-\phi_2|\leq C\varepsilon\|k-\tilde{k}\|_{L^\infty(X\times S^1\times S^1)} \tag{20}$$

with $C>0$ depending on $\mathrm{diam}(D)$ only. Then by (19), (20),

$$\|k-\tilde{k}\|_{L^\infty(X\times S^1\times S^1)}\leq C\varepsilon\|k-\tilde{k}\|_{L^\infty(X\times S^1\times S^1)},$$

and for $\varepsilon>0$ small enough this implies $k=\tilde{k}$.

It remains to prove (20). By (13),

$$\begin{aligned}\phi_2-\tilde{\phi}_2 &= (I-M)^{-1}\phi_2^{\#}-(I-\tilde{M})^{-1}\tilde{\phi}_2^{\#}\\ &= (I-M)^{-1}(M^2\phi_0-\tilde{M}^2\phi_0)+(I-\tilde{M})^{-1}(M-\tilde{M})(I-M)^{-1}\tilde{\phi}_2^{\#}\\ &= I_1+I_2.\end{aligned} \tag{21}$$

Writing $M^2-\tilde{M}^2=M(M-\tilde{M})+(M-\tilde{M})\tilde{M}$, the first term I_1 can be estimated by Lemma 1 and the paragraph after (16), thus

$$|I_1|\leq C\varepsilon\|k-\tilde{k}\|_{L^\infty}\left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right). \tag{22}$$

To estimate I_2, write

$$|I_2|\leq C\|k-\tilde{k}\|_{L^\infty}\int_{S^1}\|(I-M)^{-1}\tilde{\phi}_2\|_{L^\infty(X)}\,d\theta\leq C\varepsilon^2\|k-\tilde{k}\|_{L^\infty}. \tag{23}$$

By (21), (22), (23),

$$|\phi_2-\tilde{\phi}_2|\leq C\varepsilon\|k-\tilde{k}\|_{L^\infty}\left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right).$$

for a.e. $(x,\theta,x',\theta')\in X\times S^1\times\Gamma_-$ and for a.e. $(x,\theta,x',\theta')\in\Gamma_+\times\Gamma_-$. The last statement follows from the fact that $M=T_1^{-1}K$ and from the smoothing properties of T_1^{-1}. Since $\sin|\theta-\theta'|\log(1/\sin|\theta-\theta'|)$ is bounded, this proves (20).

This completes the proof of Theorem 1. $\square$

Remark. It follows from (19) and (20) that we can choose $\varepsilon = C(d)e^{-2d\Sigma}$, where $d = \operatorname{diam}(D)$.

2 The stability estimate, proof of Theorem 2

According to Proposition 1, for the distribution kernel $\alpha(x,\theta,x',\theta')$ of $\mathcal{A}$ we have the representation (4), where a is as in (5), while $b = (\phi_1+\phi_2)|_{(x,\theta)\in\Gamma_+}$. Then b is a function and an elementary calculation show that $b \in L^\infty(\Gamma_+, L^1(\Gamma_-))$. Proposition 1 shows that $b\sin(\theta-\theta') \in L^\infty$. Assume that we have two pairs of continuous functions (σ,k) and $(\tilde\sigma,\tilde k)$ in $\mathcal{V}^s_{\Sigma,\varepsilon}$ with albedo operators $\mathcal{A}$ and $\tilde{\mathcal{A}}$, respectively. In what follows we will denote the quantities a, b, α, etc., related to the second pair by putting a tilde sign over it. Also, we will use the notation $\Delta\mathcal{A}$ to denote the difference $\Delta\mathcal{A} = \mathcal{A}-\tilde{\mathcal{A}}$, and similarly $\Delta a = a-\tilde a$, etc. In this paper, Δ never stands for the Laplacian.

Let δ_1 and δ_2 be as in (6). According to Proposition 1, $\delta_2 < \infty$. Our goal is to estimate $\Delta\sigma$ and Δk in terms of δ_1 and δ_2. Observe that, as in the uniqueness proof, Δe^{-a} and $\Delta(\alpha_1+\alpha_2)$ can be recovered from $\Delta\alpha$ by separating the most singular part of $\Delta\alpha$ from the rest. Therefore, δ_1 measures the magnitude of the singular part of $\Delta\alpha$, while δ_2 measures the magnitude of the regular part.

We start with estimating $\Delta\sigma$. By a result of Mukhometov [M],

$$\|\Delta\sigma\|_{L^2} \le C\delta_1. \tag{24}$$

Next we estimate Δk in terms of δ_1 and δ_2. Set $E_1(y,\theta,\theta') = E(y,\theta',-\infty)E(y,\theta,\infty)$. Then by (14), $\sin|\theta-\theta'|\alpha_1(x,\theta,x',\theta') = \chi(E_1k)(y,\theta',\theta)$ with y, χ as in (14). Our starting point is the relation $\Delta(E_1k) = k\Delta E_1 + \tilde E_1\Delta k$. By the inequality $|e^{-s_1}-e^{-s_2}| \le |s_1-s_2|$ for $0 \le s_i$, $i=1,2$, we get that $|\Delta E_1| \le 2d|\Delta\sigma| \le 2d\delta_1'$, where

$$\delta_1' = \|\Delta\sigma\|_{L^\infty}.$$

Hence, on $\operatorname{supp}\chi$,

$$\begin{aligned}
\tilde E_1|\Delta k|(y,\theta,\theta') &\le |\Delta(E_1k)| + k|\Delta E_1| \\
&\le |\Delta(\alpha_1+\alpha_2)\sin|\theta-\theta'| - \Delta\alpha_2\sin|\theta-\theta'|| + C\delta_1' \\
&\le \delta_2 + |\Delta\alpha_2|\sin|\theta-\theta'| + C\delta_1'.
\end{aligned} \tag{25}$$

Therefore,

$$\|\Delta k\|_{L^\infty} \le C\left(\delta_1' + \delta_2 + \|\Delta\alpha_2\sin|\theta-\theta'|\|_{L^\infty}\right) \tag{26}$$

We want to prove an estimate similar to (20) on the last term in the r.h.s. above. Recall that in (20), we have $\sigma=\tilde\sigma$, while here we only know how to estimate the difference $\Delta\sigma = \sigma-\tilde\sigma$. Nevertheless, we can proceed along similar lines. Denote by γ the trace on Γ_+. Clearly, $\gamma : C(\bar X\times S^1) \to C(\Gamma_+)$ is bounded with norm 1. Note also that since we assume here that

σ and k are continuous, $(I-M)^{-1}$ exists in $C(X\times S^1)$ for $\varepsilon \ll 1$. Similarly to (20),

$$\begin{aligned}\Delta\alpha_2 &= \gamma(I-M)^{-1}\phi_2^{\#} - \gamma(I-\tilde{M})^{-1}\tilde{\phi}_2^{\#}\\ &= \gamma(I-M)^{-1}(M^2\phi_0 - \tilde{M}^2\tilde{\phi}_0) + \gamma(I-\tilde{M})^{-1}(M-\tilde{M})(I-M)^{-1}\tilde{\phi}_2^{\#}\\ &= I_1 + I_2. \end{aligned} \tag{27}$$

To estimate I_1, we write $M^2\phi_0 - \tilde{M}^2\tilde{\phi}_0 = (M^2-\tilde{M}^2)\phi_0 + \tilde{M}^2\Delta\phi_0$. In view of (21), (22), the contribution to I_1 of the first term in the r.h.s above can be estimated similarly to (22) with $\|\Delta k\|_{L^\infty}$ there replaced by $\|\Delta k\|_{L^\infty}+\|\Delta\sigma\|_{L^\infty}$. Next, about the contribution of $\tilde{M}^2\Delta\phi_0$ to I_1, the first formula in Proposition 1 easily implies that $|\tilde{M}^2\Delta\phi_0| = O(\delta_1')/(1-\log\sin|\theta-\theta'|)$, hence $|\gamma(I-M)^{-1}\tilde{M}^2\Delta\phi_0| = O(\delta_1')/(1-\log\sin|\theta-\theta'|)$. This shows that we have the following analogue of (22)

$$|I_1| \le C\varepsilon\left(\|\Delta k\|_{L^\infty} + \delta_1'\right)\left(1+\log\frac{1}{|\sin(\theta-\theta')|}\right). \tag{28}$$

In a similar way, we obtain the version of (23) in this case

$$|I_2| \le C\varepsilon^2\left(\|\Delta k\|_{L^\infty} + \delta_1'\right), \tag{29}$$

Combining (27), (28) and (29), we get, similarly to (20),

$$|\Delta\alpha_2 \sin|\theta-\theta'||\|_{L^\infty} \le C\varepsilon\left(\|\Delta k\|_{L^\infty} + \delta_1'\right) \tag{30}$$

Therefore, for $\varepsilon > 0$ small enough, (26) implies the estimate

$$\|\Delta k\|_{L^\infty} \le C\left(\delta_1' + \delta_2\right). \tag{31}$$

Estimate (31) is the base of our stability estimate. We are going now to use the interpolation inequality

$$\|f\|_{\alpha_1 s_1+\alpha_2 s_2} \le \|f\|_{s_1}^{\alpha_1}\|f\|_{s_2}^{\alpha_2}, \quad \alpha_1+\alpha_2 = 1,\ \alpha_j \ge 0,$$

where, in our case, $\|\cdot\|_s$ will be the norm in the Sobolev space $H^s(X)$. Let σ and $\tilde{\sigma}$ belong to $\mathcal{V}^s_{\Sigma,\varepsilon}$. Then

$$\|\Delta\sigma\|_{1+s\mu} \le C(\Sigma)\|\Delta\sigma\|_{L^2}^{1-1/s-\mu}$$

for any fixed $0\le\mu\le 1-1/s$. By a standard Sobolev embedding theorem and (24),

$$\delta_1' = \|\Delta\sigma\|_{L^\infty} \le C(\Sigma)\|\Delta\sigma\|_{L^2}^{1/s+\mu} \le C'(\Sigma)\delta_1^{1-1/s-\mu}. \tag{32}$$

Therefore, (31) yields

$$\|\Delta k\|_{L^\infty} \le C\left(\delta_1^{1-1/s-\mu} + \delta_2\right). \tag{33}$$

Estimates (32) and (33) complete the proof of Theorem 2. $\square$

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