TENSOR TOMOGRAPHY:PROGRESS AND CHALLENGES

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Abstract. We survey some recent progress on the problem of recovering a tensor from its integral along geodesics in the two dimensional case. We also propose several open problems.

1. Introduction

We define the geodesic ray transform for any compact, oriented Riemannian manifold (M, g) with boundary of any dimension. Let ν denote the unit-inner normal to ∂M. We denote by $S(M) \to M$ the unit-sphere bundle over M:

$$S(M) = \bigcup_{x \in M} S_x, \quad S_x = \{\xi \in T_x(M) : |\xi|_g = 1\}.$$

$S(M)$ is a $(2 \dim M - 1)$-dimensional compact manifold with boundary, which can be written as the union $\partial S(M) = \partial_+ S(M) \cup \partial_- S(M)$

$$\partial_\pm S(M) = \{(x, \xi) \in \partial S(M), \ \pm (\nu(x), \xi) \geq 0 \}.$$

The manifold of inner vectors $\partial_+ S(M)$ and outer vectors $\partial_- S(M)$ intersect at the set of tangent vectors

$$\partial_0 S(M) = \{(x, \xi) \in \partial S(M), \quad (\nu(x), \xi) = 0 \}.$$

The standard measures that we will use are defined below,

$$\begin{aligned} d\Sigma^{2n-1} &= dV^n \wedge dS_x \\ d\Sigma^{2n-2} &= dV^{n-1} \wedge dS_x \end{aligned}$$

where dV^n (resp. dV^{n-1}) is the volume form of M (resp. ∂M), and $dS = \sqrt{\det g(x)} dS_x$ where dS_x is the Euclidean volume form of S_x in $T_x M$. For $(x, \xi) \in \partial S(M)$, let $\mu(x, \xi) =< \nu(x), \xi >$ and $L^2_\mu(\partial_+ S(M))$ is the space of functions on $\partial_+ S(M)$ with inner product

$$(u, v)_{L^2_\mu(\partial_+ S(M))} = \int_{\partial_+ S(M)} uv\mu \, d\Sigma^{2n-2}$$

Assume that (M, g) is embedded in (S, g) where S is a compact n-dimensional manifold without boundary. Let φ_t be the geodesic flow on S and $X = \frac{d}{dt}\varphi_t|_{t=0}$ be the geodesic vector field. If $(x, v) \in S(M)$, let $\gamma(t, x, v)$ be the unit speed S-geodesic starting from x in the direction of v. Define travel time $\tau : S(M) \to [0, \infty]$ by

$$\tau(x, v) = \inf\{t > 0 : \gamma(t, x, v) \in S \backslash M\}$$

Definition 1.1. *(M, g) is non-trapping if $\tau(x, v) < \infty$ for all $(x, v) \in S(M)$.*

The function $\tau^0 = \tau|_{\partial S(M)}$ is equal zero on $\partial_- S(M)$ and is smooth on $\partial_+ S(M)$. Its odd part with respect to ξ

$$\tau_-^0(x,\xi) = \frac{1}{2}\left(\tau^0(x,\xi) - \tau^0(x,-\xi)\right)$$

is a smooth function.

Let u^f be the solution of the boundary value problem

$$Xu = -f, \quad u|_{\partial_- S(M)} = 0,$$

which can be written as

$$u^f(x,v) = \int\limits_0^{\tau(x,v)} f(\varphi_t(x,v))dt, \quad (x,v) \in S(M).$$

In particular

$$X\tau = -1.$$

The trace

$$If = u^f|_{\partial_+ S(M)}$$

is called *the geodesic ray transform* of the function f. If the manifold (M,g) is non-trapping, that is every geodesic has finite length, $I : C^\infty(S(M)) \to C^\infty(\partial_+ S(M))$.

We define $\psi : S(M) \to \partial_- S(M)$ by

$$\psi(x,v) = \varphi_{-\tau(x,-v)}(x,v), (x,v) \in S(M).$$

So, φ is the end point which maps the vector (x,v) along the geodesic $\gamma(x,v,t)$ in the back direction into an incoming vector. The solution of the boundary value problem for the transport equation

$$Xu = 0, \quad u|_{\partial_+ S(M)} = w$$

can be written in the form

$$u = w_\psi = w \circ \psi.$$

It is easy to show using Santalo's formula:

Proposition 1.2. *$I : L^2(M) \to L^2_\mu(\partial_+ S(M))$ is bounded.*

The adjoint I^* is bounded $L^2_\mu(\partial_+ S(M)) \to L^2(M)$. For $f \in C^\infty(M)$, $w \in C^\infty(\partial_+ S(M))$,

$$\begin{aligned}
(If, w)_{L^2_\mu(\partial_+ S(M))} &= \int_{\partial_+ S(M)} \int_0^{\tau(x,v)} f(\varphi_t(x,v)) w_\psi(\varphi_t(x,v)) \mu \, dt d\Sigma^{2n-2} \\
&= \int_{S(M)} f w_\psi d\Sigma^{2n-1} \\
&= \int_{S(M)} f(x) \left(\int_{S_x} w_\psi(x,v) \, dS_x(v) \right) dV^n(x)
\end{aligned}$$

so

$$I^* w(x) = \int_{S_x} w_\psi(x,v) \, dS_x$$

.

Clearly a function f is not determined by its geodesic ray transform alone, since it depends on more variables than If. We consider the geodesic ray transform acting on symmetric tensor fields.

We denote by $f_m(x, v)$ an homogeneous polynomial of degree m with respect to v, induced by the symmetric tensor field f on (M, g) of m degree :

$$f_m(x, v) = f_{i_1 \dots i_m}(x)\, v^{i_1} \dots v^{i_m}. \tag{1}$$

The operator I_m, defined by

$$I_m f = I f_m \tag{2}$$

is called *the geodesic ray transform* of the symmetric tensor field f. If the manifold (M, g) is non-trapping and the boundary ∂M is strictly convex $I_m : C^\infty(M, S_m(M)) \to C^\infty(\partial_+ S(M))$, where $S_m(M)$ denotes the bundle of symmetric tensors over (M, g).

The adjoint of the operator I_m is the bounded operator $I_m^* : L^2_\mu(\partial_+ S(M)) \to L^2(M, S_m(M))$ which is given by

$$(I_m^* w)^{i_1 \dots i_m}(x) = \int\limits_{S_x} w_\psi(x, v) v^{i_1} \dots v^{i_m} dS_x.$$

The Hilbert space $L^2(M, S_m(M))$ may be considered as subspace of $L^2(S(M))$ of homogeneous polynomials with respect to v of degree m. distributions.$I_m : L^2(S(M)) \to L^2_\mu(\partial_+ S(M))$ is given by

$$I^* w = w_\psi.$$

It is known that any symmetric smooth enough tensor field f may be decomposed in a potential and solenoidal part [16]:

$$f = dp + f^s, \ \ p|_{\partial M} = 0, \ \ \delta f^s = 0,$$

where δ notes the divergence and $d = \sigma \nabla$ is the symmetric part of the covariant derivative. It is easy to see that the geodesic ray transform of the potential part dp is zero. We denote by $C^\infty_{sol}(M, S_m(M))$ the space of smooth solenoidal symmetric tensor fields so that we can recoverer only the solenoidal part of the tensor field.

Definition 1.3. *A tensor of order m is s-injective if $I_m f = 0$ implies $f^s = 0$.*

The transforms I_m arise in several applications as well as in the boundary rigidity problem. The case of I_0 when the metric is Euclidean is the standard X-ray transform that integrates a function along lines. Radon found in 1917 an inversion formula in two dimensions to determine a function knowing the X-ray transform. This formula is non-local in the sense that in order to find the function at a point x one needs to know the integral of the function along lines far from the point. Radon's inversion formula has been implemented numerically using the filtered backprojection algorithm which is used today in CT scans. Another important transform in medical imaging and other applications is the Doppler transform which integrates a vector field along lines. This corresponds to the case of I_1 for the case of the Euclidean metric. The motivation is ultrasound Doppler tomography. It is known that blood flow is irregular and faster around tumor tissue than in normal tissue and Doppler tomography attempts to reconstruct the blood flow pattern. Mathematically the problem is to what extend a vector field is determined from its integral along lines. The case of integration along more general geodesics arises in geophysical imaging in determining the inner structure of the Earth since the speed of elastic waves generally increases with depth, thus curving the rays back to the Earth surface. It also arises in ultrasound imaging.

The geodesic ray transform I_0, that is, the integration of a function along geodesics, arises as the linearization of the problem of determining a conformal factor of a Riemannian metric on a compact Riemannian manifold with boundary from the boundary distance function. This is the boundary rigidity problem, see [20] for a recent review. the linearization of the boundary rigidity problem is I_2 the integration of tensors of order two along geodesics. The case of integration of tensors of order 4 along geodesics arises in some inverse problems arising in elasticity [16].

We assume throughout that (M, g) is *simple*, a notion that naturally arises in the context of the boundary rigidity problem [7]. We recall that a Riemannian manifold with boundary is said to be simple if the boundary is strictly convex and given any point p in M the exponential map $\exp_p$ is a diffeomorphism onto M. In particular, a simple manifold is nontrapping.

One of the main results we review in this paper is the s-injectivity of I_m for all m for simple surfaces.

Theorem 1.4. *Let (M, g) be a simple 2D manifold and let $m \geq 0$. If f is a smooth symmetric m-tensor field on M which satisfies $If = 0$, then $f = dh$ for some smooth symmetric $(m-1)$-tensor field h on M with $h|_{\partial M} = 0$. (If $m = 0$, then $f = 0$.)*

We also review what is known about stability, reconstruction and the range for I_m. We also propose several open problems.

2. Pestov Identity

In this section we consider the Pestov identity, which is the basic energy identity that has been used since the work of Mukhometov [8] in most injectivity proofs of ray transforms in the absence of real-analyticity or special symmetries. The Pestov identity often appears in a somewhat ad hoc way, but in [10] a new point of view was given which makes its derivation more transparent. We give an account of this for I_0 in the two dimensional case.

Since M is assumed oriented there is a circle action on the fibers of SM with infinitesimal generator V called the *vertical vector field*. It is possible to complete the pair X, V to a global frame of $T(SM)$ by considering the vector field $X_\perp := [X, V]$. There are two additional structure equations given by $X = [V, X_\perp]$ and $[X, X_\perp] = -KV$ where K is the Gaussian curvature of the surface. Using this frame we can define a Riemannian metric on SM by declaring $\{X, X_\perp, V\}$ to be an orthonormal basis and the volume form of this metric will be denoted by $d\Sigma^3$. The fact that $\{X, X_\perp, V\}$ are orthonormal together with the commutator formulas implies that the Lie derivative of $d\Sigma^3$ along the three vector fields vanishes.

We consider the ray transform on functions. The first step is to recast the injectivity problem as a uniqueness question for the partial differential operator P on SM where

$$P := VX.$$

This involves a standard reduction to the transport equation.

Proposition 2.1. *Let (M, g) be a compact oriented nontrapping surface with strictly convex smooth boundary. The following statements are equivalent.*

(a) *The geodesic ray transform $I_0 : C^\infty(M) \to C(\partial_+(SM))$ is injective.*
(b) *Any smooth solution of $Pu = 0$ in SM with $u|_{\partial(SM)} = 0$ is identically zero.*

Proof. Assume that the ray transform is injective, and let $u \in C^\infty(SM)$ solve $Pu = 0$ in SM with $u|_{\partial(SM)} = 0$. This implies that $Xu = -f$ in SM for some smooth f only depending on x, and we have $0 = u|_{\partial_+(SM)} = If$. Since I is injective one has $f = 0$ and thus $Xu = 0$, which implies $u = 0$ by the boundary condition.

Conversely, assume that the only smooth solution of $Pu = 0$ in SM which vanishes on $\partial(SM)$ is zero. Let $f \in C^\infty(M)$ be a function with $If = 0$, and define the function

$$u(x, v) := \int_0^{\tau(x,v)} f(\gamma(t, x, v))\, dt, \quad (x, v) \in SM.$$

This function satisfies the transport equation $Xu = -f$ in SM and $u|_{\partial(SM)} = 0$ since $If = 0$, and also $u \in C^\infty(SM)$. Since f only depends on x we have $Vf = 0$, and consequently $Pu = 0$ in SM and $u|_{\partial(SM)} = 0$. It follows that $u = 0$ and also $f = -Xu = 0$. □

We now focus on proving a uniqueness statement for solutions of $Pu = 0$ in SM. For this it is convenient to express P in terms of its self-adjoint and skew-adjoint parts in the $L^2(SM)$ inner product as

$$P = A + iB, \quad A := \frac{P + P^*}{2}, \quad B := \frac{P - P^*}{2i}.$$

Here the formal adjoint P^* of P is given by

$$P^* := XV.$$

In fact, if $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$, then

$$\begin{aligned} \|Pu\|^2 &= ((A + iB)u, (A + iB)u) = \|Au\|^2 + \|Bu\|^2 + i(Bu, Au) - i(Au, Bu) \\ &= \|Au\|^2 + \|Bu\|^2 + (i[A, B]u, u). \end{aligned} \tag{3}$$

This computation suggests to study the commutator $i[A, B]$. We note that the argument just presented is typical in the proof of L^2 Carleman estimates [**?**].

By the definition of A and B it easily follows that $i[A, B] = \frac{1}{2}[P^*, P]$. By the commutation formulas for X, $X_\perp$ and V, this commutator may be expressed as

$$\begin{aligned} [P^*, P] &= XVVX - VXXV = VXVX + X_\perp VX - VXVX - VXX_\perp \\ &= V[X_\perp, X] - X^2 \\ &= -X^2 + VKV. \end{aligned}$$

Consequently

$$([P^*, P]u, u) = \|Xu\|^2 - (KVu, Vu).$$

If the curvature K is nonpositive, then $[P^*, P]$ is positive semidefinite. More generally, one can try to use the other positive terms in (3). Note that

$$\|Au\|^2 + \|Bu\|^2 = \frac{1}{2}(\|Pu\|^2 + \|P^*u\|^2).$$

The identity (3) may then be expressed as

$$\|Pu\|^2 = \|P^*u\|^2 + ([P^*, P]u, u).$$

Moving the term $\|Pu\|^2$ to the other side, we have now proved the version of the Pestov identity which is most suited for our purposes. The main point in this proof was that the Pestov identity boils down to a standard L^2 estimate based on separating the self-adjoint and skew-adjoint parts of P and on computing one commutator, $[P^*, P]$.

Proposition 2.2. *If (M, g) is a compact oriented surface with smooth boundary, then*

$$\|XVu\|^2 - (KVu, Vu) + \|Xu\|^2 - \|VXu\|^2 = 0$$

for any $u \in C^\infty(SM)$ with $u|_{\partial(SM)} = 0$.

It is well known (cf. proof of [3, Proposition 7.2]) that on a simple surface, one has

$$\|XVu\|^2 - (KVu, Vu) \geq 0, \quad u \in C^\infty(SM),\ u|_{\partial(SM)} = 0.$$

Also, if $Xu = -f$ where $f = f_0 + f_1 + f_{-1}$ is the sum of a 0-form and 1-form, we have

$$\|Xu\|^2 - \|VXu\|^2 = \|f_0\|^2 \geq 0.$$

This term may be negative, and the Pestov identity may not give useful information unless there is some extra positivity like a curvature bound.

In the scalar case the following result holds on the solvability of $I_m^*, m = 0, 1$ [14].

Theorem 2.3. *Let (M, g) be a simple, compact Riemannian manifold with boundary. Then the operator $I_0^* : C_\alpha^\infty(\partial_+ S(M)) \to C^\infty(M)$ is onto.*

3. A microlocal approach

A different approach that is useful to prove s-injectivity of I_m in some cases and gives stability estimates as well as reconstruction formulas in some cases was started in [21] and developed further in [19],[**?**], [**?**]. We describe the result in more detail for I_0.

Theorem 3.1. *$I_0^* I_0$ is an elliptic pseudodifferential operator on S of order -1.*

therefore Fredholm and with close range. The surjectivity of I_0^* follows then since I_0 is injective.

Proof. It is easy to see, that

$$(4) \qquad (I_0^* I_0 f)(x) = \int\limits_{\Omega_x} d\Omega_x \int\limits_{-\tau(x,-v)}^{\tau(x,v)} f(\gamma(x, v, t))\, dt = 2 \int\limits_{\Omega_x} d\Omega_x \int\limits_{0}^{\tau(x,v)} f(\gamma(x, n, t))\, dt.$$

Before we continue we make a remark concerning notation. We have used up to know the notation $\gamma(x, v, t)$ for a geodesic. But it is known, that a geodesic depends smoothly on the point x and vector $\xi t \in T_x(M)$. Therefore in what follows we will also use sometimes the notation $\gamma(x, vt)$ for a geodesic. Since the manifold M is simple and any small enough neighborhood U (in (S,g)) is also simple (an open domain is simple if its closure is simple). For any point $x \in U$ there is an open domain $D_x^U \subset T_x(U)$ such that exponential map $exp_x : D_x^U \to U,\ exp_x\eta = \gamma(x,\eta)$ is a diffeomorphism onto U. Let $D_x,\ x \in M$ be the inverse image of M, then $exp_x(D_x) = M$ and $exp_x|_{D_x} : D_x \to M$ is a diffeomorphism.

Now we change variables in (3.1), $y = \gamma(x, vt)$. Then $t = d_g(x,y)$ and

$$(I^*If)(x) = \int_M K(x,y)\ f(y)\,dy,$$

where

$$K(x,y) = 2\frac{\det\left(exp_x^{-1}\right)'(x,y)\sqrt{\det g(x)}}{d_g^{n-1}(x,y)}.$$

Notice, that since

$$\gamma(x,\eta) = x + \eta + O(|\eta|^2), \tag{5}$$

it follows, that the Jacobian matrix of the exponential map is 1 at 0, and then $\det(exp_x^{-1\prime})(x,x) = 1/\det(exp_x)'(x,0) = 1$. From (3) we also conclude that

$$d^2(x,y) = G_{ij}(x,y)(x-y)^i(x-y)^j,\ \ G_{ij}(x,x) = g_{ij}(x),\ \ G_{ij} \in C^\infty(M \times M)$$

Therefore the kernel of I^*I can be written in the form

$$K(x,y) = \frac{2\det\left(exp_x^{-1}\right)'(x,y)\sqrt{\det g(x)}}{\left(G_{ij}(x,y)(x-y)^i(x-y)^j\right)^{(n-1)/2}}.$$

Thus the kernel K has at the diagonal $x = y$ a singularity of type $|x-y|^{-n+1}$. The kernel

$$K_0(x,y) = \frac{2\sqrt{\det g(x)}}{\left(g_{ij}(x)(x-y)^i(x-y)^j\right)^{(n-1)/2}}$$

has the same singularity. Clearly, the difference $K - K_0$ has a singularity of type $|x-y|^{-n+2}$. Therefore the principal symbols of both operators coincide. The principal symbol of the integral operator, corresponding to the kernel K_0 coincide with its full symbol and is easily calculated. As a result

$$\sigma(I_0^*I)(x,v) = 2\sqrt{\det g(x)}\int \frac{e^{-i(y,v)}}{\left(g_{ij}(x)y^iy^j\right)^{(n-1)/2}}dy = c_n|v|^{-1}.$$

□

The analog result for vector fields was proven in [**?**].

Theorem 3.2. *Let (M,g) be a simple, compact Riemannian manifold with boundary. Then for any field $v \in C_{sol}^\infty(M, T(M))$ there exists a function $w \in C_\alpha^\infty(\partial_+S(M))$ such*

$$v = I_1^*w$$

4. Stability estimates

Theorem 4.1. *Let g be a simple metric in M and assume that g is extended smoothly as a simple metric near the simple manifold $M_1 \supset\supset M$. Then for any function $f \in L^2(M)$,*

$$^M\|f\|/C \leq \|N_g f\|_{H^1(M_1)} \leq C\|f\|.$$

Theorem 4.2. *Assume that g is simple metric in M and extend g as a simple metric in $M_1 \supset\supset M$. Then for any 1-form $f = f_i dx^i$ in $L^2(\Omega)$ we have*

$$^M\, \|f^s\|_{L^2(M)}/C \leq \|N_g f\|_{H^1(M_1)} \leq C\,\|f^s\|_{L^2(\Omega)}.$$

Introduce the norm

$$|N_g f\|_{\tilde{H}^2(\Omega_1)} = \sum_{i=1}^n \|\partial_i N_g f\|_{\tilde{H}^1(\Omega_1)} + \|N_g f\|_{H^1(\Omega_1)}$$

The $\tilde{H}^1$ norm above is defined as in norm with the integral taken in a small two sided neighborhood of $\partial\Omega$, not only outside Ω as in norm. The norm above defines a Hilbert space $\tilde{H}^2(\Omega_1)$. We have therefore proved part (a) of the following theorem. Recall that $\mathcal{S}$ is the projection onto the space of solenoidal tensors.

Theorem 4.3. *Assume that g is simple metric in M and extend g as a simple metric in i $M_1 \supset\supset M$.*

(a) The following estimate holds for each symmetric 2-tensor f in $H^1(\Omega)$:

$$\|f^s\|_{L^2(M)} \leq C\|N_g f\|^M_{\tilde{H}^2(M_1)} + C_s\|f\|_{H^{-s}(M_1)}, \quad \forall s > 0.$$

(b) $\operatorname{Ker} I_g \cap \mathcal{S}L^2(\Omega)$ *is finite dimensional and included in $C^\infty(\bar{\Omega})$. (c) Assume that I_g is s-injective in Ω, i.e., that* $\operatorname{Ker} I_g \cap \mathcal{S}L^2(\Omega) = \{0\}$*. Then for any symmetric 2-tensor f in $H^1(\Omega)$ we have*

$$^M\,\|f^s\|_{L^2(M)} \leq C\|N_g f\|_{\tilde{H}^2(M_1)}. \tag{6}$$

5. The Scattering Relation

Suppose we have a Riemannian metric in Euclidean space which is the Euclidean metric outside a compact set. The inverse scattering problem for metrics is to determine the Riemannian metric by measuring the scattering operator (see [**?**]). obstruction to the boundary rigidity problem occurs in this case with the diffeomorphism ψ equal to the identity outside a compact set.the scattering operator, one can determine, under some non-trapping assumptions on the metric, the **scattering relation** on the boundary of a large ball. This uses high frequency information of the scattering operator. In the semiclassical setting Alexandrova has shown for a large class of operators that the scattering operator associated to potential and metric perturbations of the Euclidean Laplacian is a semiclassical Fourier integral operator quantized by the scattering relation [**?**]. The scattering relation maps the point and direction of a geodesic entering the manifold to the point and direction of exit of the geodesic. We proceed to define in more detail the scattering relation and its relation with the boundary distance function.

Definition 5.1. *Let (M,g) be non-trapping with strictly convex boundary. The scattering relation $\alpha : \partial S(M) \to \partial S(M)$ is defined by*

$$\alpha(x,v) = (\gamma(x,v,2\tau_-^0(x,\xi)), \dot\gamma(x,v,2\tau_-^0(x,v))).$$

The scattering relation is a diffeomorphism $\partial S(M) \to \partial S(M)$. Notice that $\alpha|_{\partial_+ S(M)} : \partial_+ S(M) \to \partial_- S(M)$, $\alpha|_{\partial_- S(M)} : \partial_- S(M) \to \partial_+ S(M)$ are diffeomorphisms as well. Obviously, α is an involution, $\alpha^2 = id$ and $\partial_0 S(M)$ is the hypersurface of its fixed points, $\alpha(x,\xi) = (x,\xi)$, $(x,\xi) \in \partial_0 S(M)$.

A natural inverse problem is whether the scattering relation determines the metric g up to an isometry which is the identity on the boundary. This information takes into account all the travel times not just the first arrivals.

In the case that (M,g) is a simple manifold, and we know the metric at the boundary (and this is determined if d_g is known, see [**?**]), knowing the scattering relation is equivalent to knowing the boundary distance function ([7]).

We introduce the operators of even and odd continuation with respect to α:

$$A_\pm w(x,\xi) = w(x,\xi), \qquad (x,\xi) \in \partial_+ S(M),$$
$$A_\pm w(x,\xi) = \pm (\alpha^* w)(x,\xi), \quad (x,\xi) \in \partial_- S(M).$$

We will examine next the boundness properties of A_-, A_+.

Lemma 5.2. $A_\pm : L^2_\mu(\partial_+ S(M)) \to L^2_{|\mu|}(\partial S(M))$ *are bounded.*

Proof.

$$\begin{aligned} \|A_\pm w\|^2_{L^2_{|\mu|}(\partial S(M))} &= \int_{\partial_+ S(M)} w^2 \mu \, d\Sigma^{2n-2} + \int_{\partial_- S(M)} (\alpha^* w)^2 (-\mu \, d\Sigma^{2n-2}) \\ &= \int_{\partial_+ S(M)} w^2 \mu \, d\Sigma^{2n-2} + \int_{\partial_+ S(M)} w^2 \alpha^* (-\mu \, d\Sigma^{2n-2}) \end{aligned}$$

where $\alpha : \partial_+ S(M) \to \partial_- S(M)$ is a diffeomorphism. Thus it is enough to show that

$$\alpha^*(-\mu d\Sigma^{2n-2}) = \mu d\Sigma^{2n-2}$$

Let $w \in C^\infty(\partial_+ S(M))$. Then

$$\int_{\partial_+ S(M)} w\tau\mu \, d\Sigma^{2n-2} = \int_{\partial_+ S(M)} \int_0^{\tau(x,\xi)} w_\psi(\varphi_t(x,\xi))\mu \, dt d\Sigma^{2n-2} = \int_{S(M)} w_\psi \, d\Sigma^{2n-1}$$

Set $\tilde u(x,\xi) = u(x,-\xi)$ for $u \in C^\infty(S(M))$, one has

$$\begin{aligned} \int_{S(M)} w_\psi \, d\Sigma^{2n-1} &= \int_{S(M)} \tilde w_\psi \, d\Sigma^{2n-1} \\ &= \int_{\partial_- S(M)} \int_0^{\tau(y,-\eta)} \tilde w_\psi(\varphi_t(y,-\eta))(-\mu) \, dt d\Sigma^{2n-2} \\ &= \int_{\partial_- S(M)} \int_0^{\tau(y,-\eta)} w(\alpha(y,\eta))(-\mu) \, dt d\Sigma^{2n-2} \\ &= \int_{\partial_+ S(M)} w\tau\alpha^*(-\mu d\Sigma^{2n-2}) \end{aligned}$$

Varying w shows that $\alpha^*(-\mu d\Sigma^{2n-2}) = \mu d\Sigma^{2n-2}$ on $\partial_+ S(M) \backslash \partial_0 S(M)$. ∎

The adjoint $A_\pm^*: L^2_{|\mu|}(\partial S(M)) \to L^2_\mu(\partial_+ S(M))$ satisfies

$$\begin{aligned}(A_\pm w, u)_{L^2_{|\mu|}(\partial S(M))} &= \int_{\partial_+ S(M)} wu\mu\, d\Sigma^{2n-2} \pm \int_{\partial_- S(M)} (w\circ\alpha)u(-\mu\, d\Sigma^{2n-2}) \\ &= \int_{\partial_+ S(M)} w(u \pm u\circ\alpha)\mu\, d\Sigma^{2n-2}\end{aligned}$$

so $A_\pm^* u = (u \pm u\circ\alpha)|_{\partial_+ S(M)}$.

In [14] the following characterization of the space of smooth solutions of the transport equation was given

Lemma 5.3.

$$C_\alpha^\infty(\partial_+ S(M)) = \{w \in C^\infty(\partial_+ S(M)) : A_+ w \in C^\infty(\partial S(M))\}.$$

Then $I^* w \in C^\infty(M)$ whenever $w \in C_\alpha^\infty(\partial_+ S(M))$.

6. The Hilbert Transform

We recall first the definition of the Hilbert transform on the unit disc $\partial\mathbb{D}$. Writing $x_1 + ix_2 = (x_1, x_2)$, we get $\bar\partial u = \nabla u = (\partial_1 u, \partial_2 u)$, and $-i\bar\partial v = \nabla_\perp v = (\partial_2 v, -\partial_1 v)$. Thus u and v are conjugate harmonic iff

$$\nabla u = \nabla_\perp v, \qquad \nabla v = -\nabla_\perp u$$

This is an invariant formulation, and can be used to define conjugate harmonic functions in $(T_x M, g(x)) \simeq (\mathbb{R}^2, e)$ by the following lemma (then $\nabla_\perp = \epsilon\nabla$):

Lemma 6.1. *Let M be a 2D oriented manifold. Then $\exists$aunique 2-tensor field ϵ ("multiplication by -i") such that $\{v, -\epsilon v\}$ is a positive orthonormal basis of $T_x M$ whenever $v \in T_x M$ with $|v| = 1$. It holds that*

$$< \epsilon v, \epsilon w > = < v, w >, \qquad < \epsilon v, w > = - < v, \epsilon w >$$

The Hilbert transform on $\partial\mathbb{D}$ is

$$Hf(z) = P.V. \int_{\partial\mathbb{D}} \frac{1 + Re(z\bar w)}{-Re(iz\bar w)} f(w)\, dm(w) \qquad \text{where } dm(e^{i\theta}) = \frac{1}{2\pi} d\theta$$

Write $x_1 + ix_2 = (x_1, x_2)$, then $z \cdot w = Re(z\bar w)$ and $-iz = \epsilon z = z_\perp$, so

$$Hf(z) = P.V. \int_{\partial\mathbb{D}} \frac{1 + z\cdot w}{z_\perp \cdot w} f(w)\, dm(w)$$

Let now $u \in C^\infty(S(M))$. The fiberwise Hilbert transform is defined by

$$Hu(x,\xi) = P.V. \frac{1}{2\pi} \int_{S_x} \frac{1 + <\xi, \eta>}{<\xi_\perp, \eta>} u(x,\eta)\, dS_x(\eta) \qquad \xi \in S_x. \tag{7}$$

Here $\perp$ means a 90^o degree rotation. In coordinates $(\xi_\perp)_i = \varepsilon_{ij}\xi^j$, where

$$\varepsilon = \sqrt{\det g} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The Hilbert transform H transforms even (respectively odd) functions with respect to ξ to even (respectively odd) ones. If H_+ (respectively H_-) is the even (respectively odd) part of the operator H:

$$H_+u(x,\xi) = \frac{1}{2\pi}\int_{S_x} \frac{(\xi,\eta)}{(\xi_\perp,\eta)} u(x,\eta)dS_x(\eta),$$

$$Hu_-(x,\xi) = \frac{1}{2\pi}\int_{S_x} \frac{1}{(\xi_\perp,\eta)} u(x,\eta)dS_x(\eta)$$

and u_+, u_- are the even and odd parts of the function u, then $H_+u = Hu_+, H_-u = Hu_-$. The above integrals are understood in the principal value sense.

We have that $X_\perp = (\xi_\perp, \nabla) = -(\xi, \nabla_\perp)$, where $\nabla_\perp = \varepsilon\nabla$ and ∇ is the covariant derivative with respect to the metric g. The following commutator formula for the geodesic vector field and the Hilbert transform, is a crucial ingredient in the proofs of the main theorems surveyed in these notes (see [14]).

Theorem 6.2. *Let (M,g) be a two dimensional Riemannian manifold. For any smooth function u on $S(M)$ we have the identity*

$$[H,X]u = X_\perp u_0 + X_\perp u)_0 \tag{8}$$

where

$$u_0(x) = \frac{1}{2\pi}\int_{S_x} u(x,\xi)dS_x$$

is the average value.

Proof. Using the frame $\{X, X_\perp, V\}$, defined earlier we can define a Riemannian metric on SM by declaring $\{X, X_\perp, V\}$ to be an orthonormal basis and the volume form of this metric will be denoted by $d\Sigma^3$. The fact that $\{X, X_\perp, V\}$ are orthonormal together with the commutator formulas implies that the Lie derivative of $d\Sigma^3$ along the three vector fields vanishes. Given functions $u, v : SM \to \mathbb{C}$ we consider the inner product

$$(u,v) = \int_{SM} u\bar{v}\, d\Sigma^3.$$

Since $X, X_\perp, V$ are volume preserving we have $(Vu,v) = -(u,Vv)$ for $u,v \in C^\infty(SM)$, and if additionally $u|_{\partial(SM)} = 0$ or $v|_{\partial(SM)} = 0$ then also $(Xu,v) = -(u,Xv)$ and $(X_\perp u,v) = -(u,X_\perp v)$. The space $L^2(SM)$ decomposes orthogonally as a direct sum

$$L^2(SM) = \bigoplus_{k\in\mathbb{Z}} H_k$$

where H_k is the eigenspace of $-iV$ corresponding to the eigenvalue k. A function $u \in L^2(SM)$ has a Fourier series expansion

$$u = \sum_{k=-\infty}^{\infty} u_k,$$

where $u_k \in H_k$. Also $\|u\|^2 = \sum \|u_k\|^2$, where $\|u\|^2 = (u,u)^{1/2}$. The even and odd parts of u with respect to velocity are given by

$$u_+ := \sum_{k \text{ even}} u_k, \qquad u_- := \sum_{k \text{ odd}} u_k.$$

Locally we can always choose isothermal coordinates (x_1, x_2) so that the metric can be written as $ds^2 = e^{2\lambda}(dx_1^2 + dx_2^2)$ where λ is a smooth real-valued function of $x = (x_1, x_2)$. This gives coordinates (x_1, x_2, θ) on SM whereθ is the angle between a unit vector v and $\partial/\partial x_1$. In these coordinates we may write $V = \partial/\partial\theta$ and

$$u_k(x,\theta) = \left(\frac{1}{2\pi} \int_0^{2\pi} u(x,t) e^{-ikt}\, dt \right) e^{ik\theta} = \tilde{u}_k(x) e^{ik\theta}.$$

Observe that for $k \geq 0$, u_k may be identified with a section of the k-th tensor power of the canonical line bundle; the identification takes u_k into $\tilde{u}_k e^{k\lambda}(dz)^k$ where$z = x_1 + ix_2$.

Following Guillemin and Kazhdan in [5] we introduce the first order elliptic operators $\eta_\pm : C^\infty(SM) \to C^\infty(SM)$ given by

$$\eta_+ := (X + iX_\perp)/2, \qquad \eta_- := (X - iX_\perp)/2.$$

Clearly $X = \eta_+ + \eta_-$. Let $\Omega_k := C^\infty(SM) \cap H_k$. The commutation relations $[-iV, \eta_+] = \eta_+$ and $[-iV, \eta_-] = -\eta_-$ imply that $\eta_\pm : \Omega_k \to \Omega_{k\pm1}$. If $A(x,v) = \alpha_j(x)v^j$ where α is a purely imaginary 1-form on M, we also split $A = A_+ + A_-$ where $A_\pm \in \Omega_{\pm1}$ and write

$$\mu_+ := \eta_+ + A_+, \quad \mu_- := \eta_- + A_-.$$

It suffices to show that

$$[Id + iH, X]u = iX_\perp u_0 + i(X_\perp u)_0.$$

Since $X = \eta_+ + \eta_-$, we need to compute $[Id + iH, \eta_\pm]$, so let u s find $[Id + iH, \eta_+]u$, where $u = \sum_k u_k$. Recall that $(Id + iH)u = u_0 + 2\sum_{k\geq1} u_k$. We find:

$$(Id + iH)\eta_+ u = \eta_+ u_{-1} + 2\sum_{k\geq0} \eta_+ u_k,$$

$$\eta_+(Id + iH)u = \eta_+ u_0 + 2\sum_{k\geq1} \eta_+ u_k.$$

Thus

$$[Id + iH, \eta_+]u = \eta_+ u_{-1} + \eta_+ u_0.$$

Similarly, we find

$$[Id + iH, \eta_-]u = -\eta_- u_0 - \eta_- u_1.$$

Therefore using that $iX_\perp = \eta_+ - \eta_-$ we obtain

$$[Id + iH, X]u = iX_\perp u_0 + i(X_\perp u)_0$$

as desired.

□

We define

$$P_- = A_-^* H_- A_+, \quad P_+ = A_-^* H_+ A_+.$$

Separating the odd and even parts in (1.4) we get

$$H_+\mathcal{H}u - \mathcal{H}H_-u = (\mathcal{H}_\perp u)_0, \qquad H_-\mathcal{H}u - \mathcal{H}H_+u = \mathcal{H}_\perp u_0$$

Take $u = w_\psi$ with $w \in C_\alpha^\infty(\partial_+\Omega(M))$. Then

$$2\pi\mathcal{H}H_+w_\psi = -\mathcal{H}_\perp I^*w$$

using (1) we conclude

$$2\pi A_-^* H_- A_+ w = I\mathcal{H}_\perp I^* w \tag{9}$$

since $w_\psi|_{\partial\Omega(M)} = A_+w$. Let $h = I^*w$, since $I\mathcal{H}_\perp h = I\mathcal{H}h_* = -A_-^*h_*^0$, one obtains

$$2\pi A_-^* H_- A_+ w = -A_-^* h_*^0. \tag{10}$$

7. Range and inversion of the geodesic ray transform

Let $T(M)$ be the tangent bundle of M. We denote by δ the divergence operator $\delta : C^\infty(M, TM)) \to C^\infty(M)$. In local coordinates this is given by $\delta u = g^{kj}\nabla_k u_j$ using Einstein's summation convention.

We define the operator $\delta_\perp : C^\infty(M, T(M)) \to C^\infty(M)$ by

$$\delta_\perp u = -\delta u_\perp.$$

Then

$$\delta_\perp\nabla_\perp f = \delta\nabla f = \Delta f, \;\; \delta_\perp\nabla f = -\delta\nabla_\perp f = 0.$$

We now give the characterization of the range of I_0 and I_1 in terms of the scattering relation only. We have that these are the projections of the operators P_-, P_+ respectively. For the details see [**?**].

Theorem 7.1. *Let (M, g) be simple two dimensional compact Riemannian manifold with boundary. Then*
i) The maps

$$\begin{aligned} \delta_\perp I_1^* &: C_\alpha^\infty(\partial_+S(M)) \to C^\infty(M), \\ \nabla_\perp I_0^* &: C_\alpha^\infty(\partial_+S(M)) \to C_{sol}^\infty(M, T(M)) \end{aligned}$$

are onto.
ii). A function $u \in C^\infty(\partial_+S(M))$ belong to Range I_0 iff $u = P_-w,\ w \in C_\alpha^\infty(\partial_+S(M))$.
iii). A function $u \in C^\infty(\partial_+S(M))$ belong to Range I_1 iff $u = P_+w,\ w \in C_\alpha^\infty(\partial_+S(M))$.

Proposition 7.2. *The operator $W : C_0^\infty(M) \to C^\infty(M)$, defined by*

$$Wf = (\mathcal{H}_\perp u^f)_0$$

can be extended to a smoothing operator $W : L^2(M) \to C^\infty(M)$.

We remark that in the case of constant Gaussian curvature $W = 0$ and this does not depend on whether the metric has conjugate points so that the inversion formulas of Theorem 5.2 hold for all two dimensional manifolds with boundary with constant curvature.

The inversion formulas are (see [**?**])

Theorem 7.3. *Let (M, g) be a two-dimensional simple manifold. Then we have*

$$\begin{aligned} f + W^2 f &= \frac{1}{2\pi}\delta_\perp I_1^* w, \quad w = \frac{1}{2}\alpha^* H(I_0 f)^-|_{\partial_+ S(M)}, \quad f \in L^2(M), \\ h + (W^*)^2 h &= \frac{1}{2\pi} I_0^* w, \quad w = \frac{1}{2}\alpha^* H(I_1 \mathcal{H}_\perp h)^+|_{\partial_+ S(M)}, \; h \in H_0^1(M), \end{aligned}$$

where $W, W^* : L^2(M) \to C^\infty(M)$. *In the case of a manifold of constant curvature* $W = 0, \quad W^* = 0$.

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